Solid models and B-REP
Classical modelling problem : the intersection

3 independent representations of the intersection :
- a 3D NURBS curve (giving points in the global XYZ coordinate system)
- a 2D NURBS curve in the parametric space of surface A (giving 2D points in the coordinate system of the parametric space of surface A)
- Idem for surface B
Theoretically, these three representations are equivalent ...

In practice, there are numerical approximations

- NURBS are finite approximation spaces; therefore approximation/interpolation errors do occur.
- The use of floating point numbers with a finite binary representation of the mantissa lead to numerical errors

There is no robust way to ensure, in a geometrical sense, that a curve located on surface A is the same as the corresponding curve on surface B, i.e. that both surfaces are neighbours, and share the same edge.
Definition of a **topology**: non geometric relations between entities.

- This allows to unify the calculations (of points, normals, etc...) on entities shared (or bounding) other entities (e.g. an edge shared by surfaces).
- It also allows the explicit definition of volumes – from the surfaces that bound the volume.
- It may also solve the problem of orientation of surfaces.
- Topology

B-REP model
B-Rep model

- « Boundary representation »
- Model based on the representation of surfaces
- Model of exchange (STEP format) and definition
- The “natural” set of operators is richer than for CSG
  - Extrusion, chamfer etc ...
- Does not carry the history of construction of the model (whereas CSG usually does)
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Solid modelling

- B-Rep model
  - Consists of two types of information:
    - Geometric
      - Geometric information is used for defining the spatial position, the curvatures, etc...
      - That's what we have seen until now – NURBS curves and surfaces!
    - Topological
      - This allows to make links between geometrical entities.
  - Two types of entities
    - Geometric entities: (volume), surface, curve, point
    - Topological entities: solid, face, edge, vertex
  - A topological entity "lies on" a geometric entity, which is its geometrical support (when existing)
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Solid modelling

- **B-Rep model**
  - Complete hierarchical model

- If it exists, the geometric support (G.S.) is expressed in parametric coordinates $u,v,t$.
- G.S. expressed in cartesian coordinates $x,y,z$.
- Allow to orient faces or volumes.

Diagram:
- **Solid** to **face**: 1:n
- **face** to **boundary face**: 1:n
- **face** to **boundary edge**: 1:n
- **boundary edge** to **boundary vertex**: 1:n
- **vertex** to **boundary vertex**: 1:n
- **Solid** to **boundary edges**: 1:{1 if manifold, 2 otherwise}
- **face** to **boundary face**: 1:{2 if manifold, n otherwise}
- **face** to **boundary edge**: 1:{2 if manifold, n otherwise}
- **edge** to **boundary edge**: 1:2
- **is related to**: 1:n
- **Composed of**: 1:2

- **u,v,t**
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Solid modelling
Face 1.
the geometric support is surface 1, which is a surface of revolution around the axis (Oz).

Definition of the face 1 in the parametric space of the surface 1

Edge 1.
the geometric support is the curve_xyz 2.
In the parametric space of the surface 1, corresponds to two boundary edges: 1 et 11

Definition of the edge 1 in the parametric space of the curve 1

Boundary vertex 1
Point_t t_1

Boundary vertex 2
Point_t t_2

Boundary edge 1.
the geometric support is a straight line (curve_uv 1) in the parametric space of the surface 1

Definition of the face 1 in the parametric space of the surface 1

Boundary edge corresponding to a degenerated edge
(zero length and identical points of departure and arrival: vertex 2)
Excerpt of the B-REP topology of the propeller
Links between the B-REP topology and the actual geometry of the propeller

**Face 1:**
- Loop 1
- Loop 2
- Loop 3
- Loop 4
- G.S. : Surface_xyz 1

**Surface_xyz 1:**
- Surface of revolution
  - NURBS Surface
  - Application \((u,v) \rightarrow (x,y,z)\)

**Curve_uv 1:**
- Straight line
  - Application \((t) \rightarrow (u,v)\)

**Curve_xyz 2:**
- NURBS Curve
  - Application \((t) \rightarrow (x,y,z)\)

**Boundary edge 1**
- Edge 1
- G.S. : Curve_uv 1
- G.S. : Curve_xyz 2

**Boundary vertex 1**
- Boundary vertex 1
- Boundary vertex 2
- G.S. : Curve_xyz 2

**Point_t 1**
- \(t = t_1\)

**Point_xyz 2**
- \(x = x_1, y = y_1, z = z_1\)
How to obtain the \((x,y,z)\) coordinates of the encircled point?

1 – Use the 3D vertex directly \((\text{Point}_\text{xyz} \ xxx)\)

2 – Use the boundary vertices for every 3D edge (there are 3 such edges \()\ (\text{Point}_\text{t} \ t1,t2,t3)\)
Then use those \((t)\) to get \((x,y,z)\) by the 3D edges

3 – Use the boundary vertices of the 2D boundary edges in the face (there are 2 faces, so 4 of them), \((\text{Point}_\text{t} \ t'1,t2,t'3,t'4)\). Then use those \((t)\) to obtain coordinates \((u,v)\) in the parametric space of the face, thanks to 2D curves, finally, use those \((u,v)\) to obtain \((x,y,z)\) thanks to the geometry of the face.

So there exists 8 different ways. Nothing indicates that the 8 set of 3D coordinates are exactly equal (there are numerical approximations). Only topology allows us to say that those 8 points are all referring to the same point ... at least conceptually.
B-Rep model

Euler characteristic for polyhedra
\[ \chi(S) = v - e + f \]

Euler – Poincaré formula
\[ \chi(S) = v - e + f - r = 2(s - h) \]

with

- \( v \) = number of vertices
- \( f \) = " of faces
- \( e \) = " of edges
- \( s \) = " of solids (independent volumes)
- \( h \) = " of holes – going through (topol. gender)
- \( r \) = " of internal loops (ring)
Euler's formula

**Euler characteristic**

Example: Cube

\[ v - e + f = \kappa \]

Opened and flattened Cube

\[ v - e + f = \kappa - 1 \]

Step 0: we take a face off the polyhedron and flatten it to obtain a plane graph
Euler's formula

$$v - e + f = \kappa - 1$$

Step 1: Repeat the following operation:
For each non triangular face, add one edge linking non related vertices. Each time, the number of edges and faces is increased by 1. This is repeated until no non triangular faces remain.
Euler's formula

\[ v - e + f = \kappa - 1 \]

Step 2: One alternates between these two operations
- Preferentially, delete triangles that have 2 boundary edges. Every time, \( e \) decreases by 2 and \( f \) and \( v \) by 1.
- Then, delete triangles with only one boundary edge. Each tile, \( e \) and \( f \) decrease by 1. This until only one triangle remain.
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Euler's formula

\[ v - e + f = \kappa - 1 \]
Every polygon can be decomposed into triangles. Therefore, by applying the three operations described in the previous slides, we can transform the planar graph into a triangle without changing Euler's characteristic. The triangle satisfies obviously

$$v - e + f = \kappa - 1$$

with $$\kappa - 1 = 1$$

Therefore, the planar graph verifies the formula.

So the initial polyhedron satisfies:

$$v - e + f = \kappa = 2$$
Necessity to take “rings” into account - inside faces

\[ \chi(S) = 2 \]
\[ \chi(S) = v - e + f \]
\[ \chi(S) = 8 - 12 + 6 = 2 \]

\[ \chi(S) = 12 - 17 + 7 = 2 \quad \text{OK} \]

\[ \chi(S) = 12 - 16 + 7 \neq 2 \quad \text{Not OK!} \]

Contribution of the ring
Necessity to take “holes” into account

\[ \chi(S) = 2 \]
\[ \chi(S) = v - e + f \]
\[ \chi(S) = 8 - 12 + 6 = 2 \]

“Warped” polyhedron

Not an edge!

Polyhedron with one hole, 4 edges less, 2 faces less, 4 vertices less

\[ \chi(S) = 4 - 8 + 4 \neq 2 \]

Not OK!

Contribution of the hole

\[ \chi(S) = v - e + f - r + 2h = 2 \]
Every B-rep model is identifiable (topologically) to a «point» in a 6-dimensional vector space.

- Vector space of coordinates \( v, e, f, s, h, r \).

Any topologically valid model shall verify the Euler-Poincaré relation

- This relation defines an «hyperplane» (of dimension 5) in a 6-dimensional space
- The equation of this hyperplane is:

\[
v - e + f - 2s + 2h - r = 0
\]
We can update a valid solid and modify the 6 numbers characterising a model with a transformation that yields a valid solid for which:

\[ v - e + f - 2s + 2h - r = 0 \]

- In this way, add a vertex \( (\Delta v = 1) \) must be accompanied, one way or another, by addition of an edge \( (\Delta e = 1) \) OR of the withdrawal of a face \( (\Delta f = -1) \), etc...
- Elementary operations satisfying the Euler-Poincaré relation are called Euler operators.
- They allow staying on the « hyperplane » of validity while changing the topological configuration.
Euler operators

- The use of Euler operators guarantees the *topological* validity of the result.
- Here we don't check the *geometric* validity (self-intersections etc...)
- We identify them under the form: \( \text{MaKb} \) where \( M = \text{Make} \) \( K = \text{Kill} \) and \( a \) and \( b \) are a sequence of entities: vertex, edge, face, solid, hole or ring.
  - In total, there are 99 Euler operators aiming to modify the number of entities by **at most** one unit.
    - These are divided in 49 + 49 inverses, plus the identity operator.
  - Among those 49 operators, we can chose 5 linearly independent operators (the hyperplane has 5 dimensions)
  - Those 5 independent operators form a base for the hyperplane of topologically admissible models.
Example of a set of Euler operators

- **MEV**, Make an Edge and a Vertex
- **MEF**, Make an Edge and a Face
- **MEKR**, Make an Edge and Kill a Ring
- **MVFS**, Make a Vertex, a Face and a Shell
- **KFMRH**, Kill a Face Make a Ring and a Hole

Proof by Mäntylä (1984) that those operators allow to build every valid solid (since they are independent)

Those operators form a base of the space of valid configurations (the « hyperplane »)
There are three types of operators in this set:

- **Skeleton operators** MVFS and KVFS
  - Allow to build/destroy elementary volumes

- **Local operators** MEV, KEV, MEF, KEF, KEMR, MEKR
  - Allow to modify connectivities for existing volumes
  - Don't modify fundamental topological characteristics of the surfaces - nb of handles/holes (topological gender) and number of independent volumes

- **Global operators** KFMRH and MFKRH
  - Allow to add/remove “handles” (change the topological gender)

- Only the skeleton and global operators do change the topological gender.
Euler operators

« Skeleton » operators

MVFS; KVFS

- Allow to « build » an « elementary » volume from void (which is an admissible topological structure) – or destroy it.

\[
\begin{align*}
\text{Nihil:} & \\
v &= 0 \\
e &= 0 \\
f &= 0 \\
h &= 0 \\
r &= 0 \\
s &= 0 \\
\text{MVFS:} & \\
v &= 1 \\
e &= 0 \\
f &= 1 \\
h &= 0 \\
r &= 0 \\
s &= 0 \\
\text{KVFS:} & \\
v &= 0 \\
e &= 0 \\
f &= 0 \\
h &= 1 \\
r &= 0 \\
s &= 1
\end{align*}
\]

Only one face - its boundary is reduced to a single vertex.
Euler operators
- Local operators
  MEV, KEV (case 1)
Euler operators

Local operators

$\text{MEV, KEV (case 2)}$


- Euler operators
  - Local operators

MEV, KEV (case 3)

\[
\begin{align*}
v &= 1 \\
e &= 0 \\
f &= 1 \\
h &= 0 \\
r &= 0 \\
s &= 1
\end{align*}
\]

\[
\begin{align*}
v &= 2 \\
e &= 1 \\
f &= 1 \\
h &= 0 \\
r &= 0 \\
s &= 1
\end{align*}
\]
- Euler operators
  - Local operators
    - MEF, KEF (case 1)
Euler operators

- Local operators

**MEF, KEF** (case 2)
Euler operators

- Local operators

MEF, KEF (case 3)
- Euler operators
  - Local operators

KEMR, MEKR (case 1)
Euler operators

Local operators

\( \text{KEMR, MEKR} \) (case 2: the loop for the internal ring is reduced to a single vertex)
Euler operators

Local operators

KEMR, MEKR (case 3: both loops are reduced to one vertex – one is the ring; the other is the external loop of the face)
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Solid modelling

- Euler operators
  - Global operators
    - KFMRH, MFKRH (case 1: allow the creation / destruction of holes in a solid)

\[
\begin{align*}
v &= 16 \\
e &= 24 \\
f &= 11 \\
h &= 0 \\
r &= 1 \\
s &= 1 \\

v &= 16 \\
e &= 24 \\
f &= 10 \\
h &= 1 \\
r &= 2 \\
s &= 137
\end{align*}
\]
Euler operators

Global operators

**KFMRH, MFKRH** (case 2): join two independent solids: here more judiciously called **Kill Face, Solid and Make Ring (KFSMR)**

Interpretation of global operators is sometimes confusing

\[
\begin{align*}
\begin{cases}
v = 16 \\
e = 24 \\
f = 12 \\
h = 0 \\
r = 0 \\
s = 2
\end{cases}
\end{align*}
\]
Example of use of Euler operators

<table>
<thead>
<tr>
<th>MVFS</th>
<th>MEV</th>
<th>2 x MEV</th>
<th>MEF</th>
</tr>
</thead>
<tbody>
<tr>
<td>(v=1)</td>
<td>(v=2)</td>
<td>(v=4)</td>
<td>(v=4)</td>
</tr>
<tr>
<td>(e=0)</td>
<td>(e=1)</td>
<td>(e=3)</td>
<td>(e=4)</td>
</tr>
<tr>
<td>(f=1)</td>
<td>(f=1)</td>
<td>(f=1)</td>
<td>(f=2)</td>
</tr>
<tr>
<td>(h=0)</td>
<td>(h=0)</td>
<td>(h=0)</td>
<td>(h=0)</td>
</tr>
<tr>
<td>(r=0)</td>
<td>(r=0)</td>
<td>(r=0)</td>
<td>(r=0)</td>
</tr>
<tr>
<td>(s=1)</td>
<td>(s=1)</td>
<td>(s=1)</td>
<td>(s=1)</td>
</tr>
</tbody>
</table>
Example of use of Euler operators

\[
\begin{align*}
\text{v} &= 4  \\
\text{e} &= 4  \\
\text{f} &= 2  \\
\text{h} &= 0  \\
\text{r} &= 0  \\
\text{s} &= 1
\end{align*}
\]

\[
\begin{align*}
\text{v} &= 8  \\
\text{e} &= 8  \\
\text{f} &= 2  \\
\text{h} &= 0  \\
\text{r} &= 0  \\
\text{s} &= 1
\end{align*}
\]

\[
\begin{align*}
\text{v} &= 8  \\
\text{e} &= 9  \\
\text{f} &= 3  \\
\text{h} &= 0  \\
\text{r} &= 0  \\
\text{s} &= 1
\end{align*}
\]

\[
\begin{align*}
\text{v} &= 8  \\
\text{e} &= 12  \\
\text{f} &= 6  \\
\text{h} &= 0  \\
\text{r} &= 0  \\
\text{s} &= 1
\end{align*}
\]
Example of use of Euler operators
Those operators have a vectorial form in the basis of elementary entities

\[ \begin{align*}
  &v \quad e \quad f \quad h \quad r \quad s \\
  &\begin{pmatrix} 1, 1, 0, 0, 0, 0 \end{pmatrix} \text{ – MEV, Make an Edge and a Vertex} \\
  &\begin{pmatrix} 0, 1, 1, 0, 0, 0 \end{pmatrix} \text{ – MEF, Make a Face and an Edge} \\
  &\begin{pmatrix} 0, -1, 0, 0, 1, 0 \end{pmatrix} \text{ – KEMR, Kill an Edge Make a Ring} \\
  &\begin{pmatrix} 1, 0, 1, 0, 0, 1 \end{pmatrix} \text{ – MVFS, Make a Vertex, a Face and a Solid} \\
  &\begin{pmatrix} 0, 0, -1, 1, 1, 0 \end{pmatrix} \text{ – KFMRH, Kill a Face, Make a Ring and a Hole} \\
\end{align*} \]

In order to have a complete basis of the configuration space, a vector orthogonal to the hyperplane of acceptable configurations must be added

\[ v - e + f - 2s + 2h - r = 0 \]

The coefficients of the equation of hyperplane are precisely the coordinates of the orthogonal vector...

\[ \begin{align*}
  &v \quad e \quad f \quad h \quad r \quad s \\
  &\begin{pmatrix} 1, -1, 1, 2, -1, -2 \end{pmatrix} \text{ – Euler-Poincaré} \\
\end{align*} \]
Any transformation can thus be expressed easily using matrix operations

- \( \mathbf{A} \) is a basis of the topological configurations space
- The columns of \( \mathbf{A} \) are the variation of the number of entities for each operator, and the E-P relation.

\[
\mathbf{A} = \begin{pmatrix}
1 & 0 & 0 & 1 & 0 & 1 \\
1 & 1 & -1 & 0 & 0 & -1 \\
0 & 1 & 0 & 1 & -1 & 1 \\
0 & 0 & 0 & 0 & 1 & 2 \\
0 & 0 & 1 & 0 & 1 & -1 \\
0 & 0 & 0 & 1 & 0 & -2 \\
\end{pmatrix}
\]

\[
\mathbf{q} = \mathbf{A} \cdot \mathbf{p}
\]

- Column corresponding to Euler-Poincaré's relation
- Columns corresponding to each of the Euler operators
- Vector representing the number of times that each operator is applied
- Vector representing the number (or the variation of the number) of elementary entities
A is composed of linearly independent vectors, thus one can get the inverse...

\[ \begin{align*}
q &= A \cdot p \\
A^{-1} \cdot q &= A^{-1} \cdot A \cdot p \\
p &= A^{-1} \cdot q
\end{align*} \]

They are the Euler Coordinates

A Vector that is orthogonal to the « hyperplane »...
Determination of elementary operations ...

\[ q = \begin{pmatrix} v=16 \\ e=24 \\ f=11 \\ h=0 \\ r=1 \\ s=1 \end{pmatrix} = (16, 24, 11, 0, 1, 1) \]

\[ p = A^{-1} \cdot q \]

\[ p = (15, 10, 1, 1, 0, 0)^T \]

15 x MEV, Make an Edge and a Vertex
10 x MEF, Make a Face and an Edge
1 x MVFS, Make a Vertex, a Face and a Solid
1 x KEMR, Kill an Edge Make a Ring
0 x KFMRH, Kill a Face, Make a Ring and a Hole

The vector \( q \) respects the Euler-Poincaré relation.
Are Euler coordinates sufficient to define the topology of a solid? → No.

- 7 x MEV, Make an Edge and a Vertex
- 7 x MEF, Make an Edge and a Face
- 1 x MVFS, Make a Vertex, a Face and a Solid
- 0 x KEMR, Kill an Edge Make a Ring
- 0 x KFMRH, Kill a Face, Make a Ring and a Hole

Identical Euler coordinates
Each Euler operator takes a certain number of parameters, in principle the entities to destroy/ or replace, and data necessary to creation of new entities.

These depend on the structure of data used to represent the B-Rep object.

Application of an Euler operator is not always possible, the entities involved must exist and respect some conditions.

KEF for example may only be applied on an edge separating two distinct faces... if not, one does not remove any face from the model!
### Conditions of application of Euler operators

<table>
<thead>
<tr>
<th>Operator</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>MEV, Make an Edge and a Vertex</td>
<td>Empty space</td>
</tr>
<tr>
<td>KEV, Kill an Edge and a Vertex</td>
<td>The edge has two distinct vertices</td>
</tr>
<tr>
<td>MEF, Make an Edge and an Face</td>
<td>Vertices belonging to the same boundary loop of one face</td>
</tr>
<tr>
<td>KEF, Kill an Edge and a Face</td>
<td>Distinct faces located on both sides of the edge</td>
</tr>
<tr>
<td>MEKR, Make an Edge and Kill a Ring</td>
<td>Vertices belong to distinct boundary loops of the same face</td>
</tr>
<tr>
<td>KEMR, Kill an Edge and Make a Ring</td>
<td>Same face located on both sides of the edge, which is not part of a ring</td>
</tr>
<tr>
<td>MVFS, Make a Vertex, a Face and a Shell</td>
<td>Empty space</td>
</tr>
<tr>
<td>KVFS, Kill a Vertex, a Face and a Shell</td>
<td>The shell (solid) has no edges and has only one vertex (elementary volume)</td>
</tr>
<tr>
<td>KFMRH, Kill Face Make a Ring and a Hole</td>
<td>The face cannot hold any ring</td>
</tr>
<tr>
<td>MFKRH, Make a Face, Kill a Ring and a Hole</td>
<td>May be only applied to a ring</td>
</tr>
</tbody>
</table>
Some examples of the application of Euler operators (not shown here)

- Extrusion of a face
- Junction of two solids
- Cutting out a solid by a plane
- Boolean operations between solids
Solid modelling

- The most used data structure in a manifold B-Rep representation: Half-Edge data structure
Basic entities

- **shell** contains:
  - Solid number
  - Reference to *face*, *edge*, *vertex* of solid

- **face** contains:
  - Face number
  - Ref. to an external *loop*
  - Ref. to a list of internal *loop*
  - Ref. to *shell*
  - Ref. to surface – nurbs or other – (the geometric support)
- loop contains:
  - Ref. to a list of halfedge
  - Ref. to face

- edge contains:
  - Ref. to halfedge of straight line
  - Ref. To the left halfedge
  - Ref. to a curve – nurbs or other – (the geometric support)

- halfedge contains:
  - Ref. to the parent edge
  - Ref. to the starting vertex
  - Ref. to the holding loop
• *vertex* contains:
  • Vertex number
  • Reference to one of the halfedges
  • Coordinates (the geometric support)

• A simplified implementation (without geometry other than vertices coordinates) in C++ of a B-rep modeller based on these ideas is available:

http://www.cs.utah.edu/~xchen/euler-doc/
B-Rep model

Possibility of automatic topological operations

Here, elimination of small features in order to generate a mesh for numerical simulation in mechanical engineering.
Bibliographic note

M. Mäntylä, An Introduction to Solid Modelling, Computer Science Press, 1988

I. Stroud, Solid Modelling and CAD Systems : How to Survive a CAD System, Springer, 2011 (available on-line from the university campus)