1er Master Ingénieur Mecanique & Aérospatial — Monday, January 9th, 2012 – 2 pm-5:30 pm

CAD exam — Total duration : 3:30 hours.

Course notes and slides are NOT allowed. Calculators are OK. The wording of all exercises should be returned to the instructor.

1) B-REP modeling

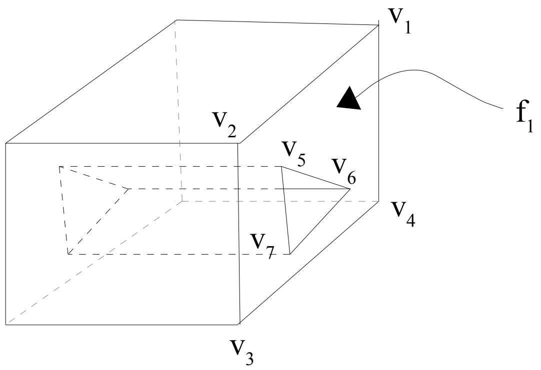


Figure 1: A solid with a hole through it.

a) A solid S is shown in figure 1 and it is asked to describe completely the B-REP representation of the face f_1. You should list the topological and geometrical entities as seen in the course, i.e. point, vertex, face, surface, edge, curve ... and draw incidence relation between these entities.
b) Give the number of topological entities in this solid (vertices, edges, faces, inner loops (rings), holes and independent solids) and check the Euler-Poincaré relation $\chi(S)=v-e+f-r=2(s-h)$.
c) One wants to use Euler operators to build the topology of this solid from scratch. Here, we list 5 operators we wish to use :

v	e	f	h	r	s	
(1,	1,	0,	0,	0,	0)	– MEV, Make an Edge and a Vertex
(0,	1,	1,	0,	0,	0)	– MEF, Make a Face and an Edge
(0,	-1,	0,	0,	1,	0)	– KEMR, Kill an Edge Make a Ring
(1,	0,	1,	0,	0,	1)	– MVFS, Make a Vertex, a Face and a Solid
(0,	0,	-1,	1,	1,	0)	– KFMRH, Kill a Face, Make a Ring and a Hole

These form an incomplete basis of the topological configurations space (only 5 basis vectors), since they only represent admissible solids with respect of the Euler-Poincaré relation. Give a complete basis **A** , by adding a vector **V** to the above matrix.

d) Using the previous basis, give a plausible sequences of operators that may be used to build the solid from void (empty space). Note that :

$$\mathbf{A}^{-1}=\frac{1}{12}\begin{pmatrix} 9 & -5 & 3 & 2 & -2 & \vdots \\ 3 & 5 & -3 & -2 & 2 & \vdots \\ -3 & 7 & 3 & 2 & -2 & \alpha\cdot\mathbf{V} \\ -6 & 2 & -6 & 4 & 8 & \vdots \\ 3 & 5 & 9 & -2 & 2 & \vdots \\ -6 & -2 & -6 & 8 & 4 & \vdots \end{pmatrix}$$

where **V** is the additional vector asked for in c) , and α is scaling parameter. In the case of your particular choice of **A**, what is the value of α ?

2) Parametrization

One gives three control points $P_i \ , i=\{0,1,2\}$ corresponding to a Bézier curve $P(u)$ in the x-y plane. It is asked here :

a) to give the expression of $P(u)$ in terms of the control points using Bernstein polynomials.

b) to give the expression of a unit tangent vector T(u)

c) to give the expression of a unit normal vector N(u)

d) to give the expression of the radius of curvature in terms of T(u) and N(u) (and/or its derivatives)

e) to compute numerically all the above for the following particular values :

$P_0=\begin{pmatrix}0\\0\end{pmatrix}, P_1=\begin{pmatrix}1\\0\end{pmatrix}, P_2=\begin{pmatrix}1\\1\end{pmatrix}$, for $u=0$ and $u=\frac{1}{2}$

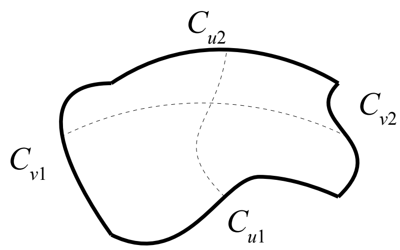
Hints : Frenet's equations may be used to compute the curvature, but beware that s is a natural parameter here:

$$\begin{bmatrix}\frac{d\,\mathrm{T}(s)}{ds}\\ \frac{d\,\mathrm{N}(s)}{ds}\end{bmatrix}=\begin{bmatrix}0 & \kappa(s)\\ -\kappa(s) & 0\end{bmatrix}.\begin{bmatrix}\mathrm{T}(s)\\ \mathrm{N}(s)\end{bmatrix}$$

Bernstein polynomials : $B_i^d(u)=\begin{pmatrix}d\\i\end{pmatrix}u^i(1-u)^{d-i}$ with $\begin{pmatrix}d\\i\end{pmatrix}=\frac{d\,!}{(d-i)!\,i\,!}$

3) Coons patch

Four compatible curves are given (*i.e.* the extremities and parametrizations match). Those are called C_{u1} and C_{u2} (parameter u) and C_{v1} and C_{v2} (parameter v). In the sequel, u and v are between 0 and 1.



One wishes to build a bilinear Coons patch $S_c(u,v)$ that is supported by (and interpolates) the four curves.

a) what is the expression of the patch in terms of the curves $C_{u1}(u)$, $C_{u2}(u)$, $C_{v1}(v)$, $C_{v2}(v)$?

The four curves are in fact Bézier curves of degree 2. They are described by three control points :

$C_{u1}(u)$: $P^0_{u1}, P^1_{u1}, P^2_{u1}$ $C_{u2}(u)$: $P^0_{u2}, P^1_{u2}, P^2_{u2}$ $C_{v1}(v)$: $P^0_{v1}, P^1_{v1}, P^2_{v1}$ $C_{v2}(v)$: $P^0_{v2}, P^1_{v2}, P^2_{v2}$

b) Express the Coons patch in terms of a Bézier surface. Coordinates of the control points P^{ij}_s will be given in terms of the control points of the 4 curves.

Hint : Forrest's degree elevation formulas may be useful :

$$Q^0=P^0$$

$$Q^i=\frac{i}{d+1}P^{i-1}+(1-\frac{i}{d+1})P^i \text{ for } i=1,\cdots,d$$

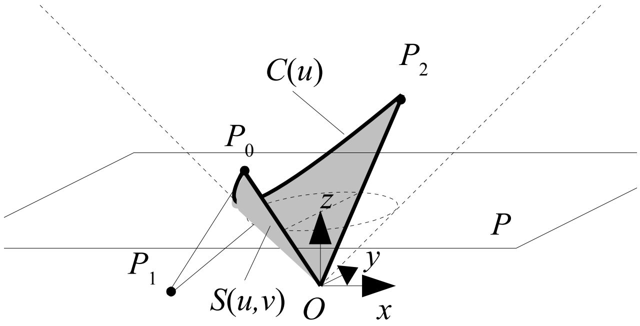
$$Q^{d+1}=P^d$$

4) Surface modeling and conics

Three control points of a degree two Bézier curve $C(u)$ are given :

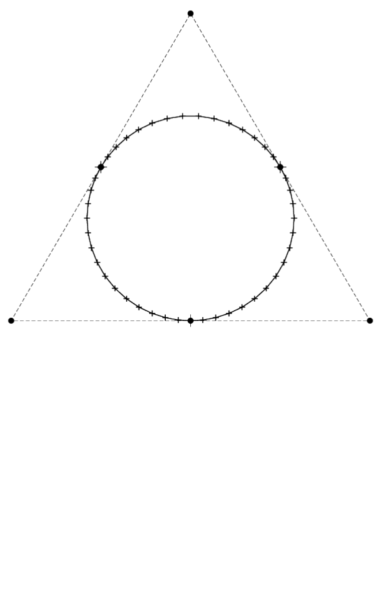
$$P_0=\begin{pmatrix}0\\-2\\2\end{pmatrix},\ P_1=\begin{pmatrix}-2\\0\\0\end{pmatrix},\ P_2=\begin{pmatrix}0\\2\\2\end{pmatrix}$$, and the origin of the frame is $O=\begin{pmatrix}0\\0\\0\end{pmatrix}$.

We wish to build a conical surface $S(u,v)$ based on this curve. The apex should be the origin O. We have $S(u,0) = O$ for all u, and $S(u,1) = C(u)$.



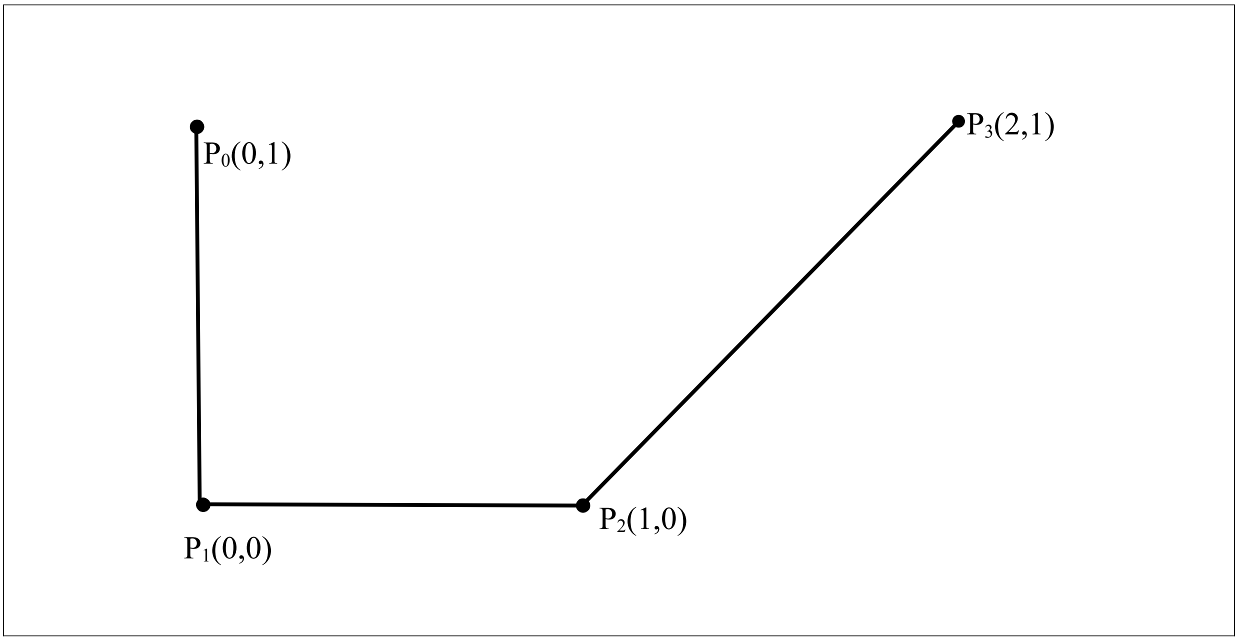
a) What is the analytical expression of the surface $S(u,v)$?
b) Express this surface in terms of a B-Spline surface : give the nodal sequences and the control points.
Now one wants to do a central projection of the curve C onto the plane P whose equation is $z - 1 = 0$. The resulting curve is C^*
c) What is the analytical expression of the new curve $C^*(u)$? What is the nature of this curve ? Prove it.
d) Give an analytic expression in the parametric space of $S(u,v)$ of the curve $C^*(u)=S(U(t),V(t))$, *i.e.* the functions $U(t)$ and $V(t)$.
e) Show that, using homogeneous coordinates, one can represent the new curve $C^*(u)$ using a Bézier or a B-Spline curve, *i.e.* using a polynomial approximation. What are the coordinates in the homogeneous space of its control points ?

f) (this question is independent from above)
A circle is represented by 3 arcs using one NURBS, as in the following drawing.
 i) Give the minimal (shortest) nodal sequence achieving this.
 ii) What is the geometrical continuity G_α of the NURBS ? What is its parametric continuity C_β ?



5) B-Spline curves

We focus on a B-Spline curve of continuity *at least* C_0, for which the control polygon is given. The nodal sequence is non uniform (therefore with a certain number of repetitions at the beginning and the end to ensure interpolation of the first and last control points – however, interior nodes are set at regular intervals). The parameter u varies between 0 and 1.



a) What are the admissible values for the degree ? Give the nodal sequence and specify the degree of continuity of the curve for each case ?
b) The degree is now 2. Give an example of non uniform nodal sequence ensuring that the curve interpolates P_3, whatever its position. What is the continuity of the curve in that case ?
c) Now , the degree is set to its maximal value. What particular curve do we obtain ?
d) Give formally De Casteljau's algorithm
e) Give the coordinates of the point corresponding to u=1/3 using De Casteljau's algorithm.
f) How to obtain the control polygon of the hodograph of the 1st derivative of the curve ? Give the formal algorithm if one wants to compute the derivative instead of the position.

6) Generalities

Fill **each** cell in the matrix with *yes* or *no* . A bad answer cancel points, therefore it is advised to respond only if you are sure of your answer – (no answer does not mean "*no*")

	Lagrange polynomials	Natural Splines	Cardinal Splines	Bézier curves	B-splines curves	NURBS curves
Allow "periodic" curves						
Interpolates the control points						
Allows local control						
Has a polynomial expression (eventually by parts)						
Allows the exact representation of conics						
Allows arbitrary discontinuities in the derivatives						
Allows the exact representation of an "offset" curve						
Is «variation diminishing»						