1er Master Ingénieur Mécanique & Aérospatial — Monday, January 7th, 2013 – 2 pm-5:30 pm
CAD exam — Total duration : 3:30 hours.

Course notes and slides are NOT allowed. Calculators are OK. The wording of all exercises should be returned to the instructor.

1) B-REP modeling

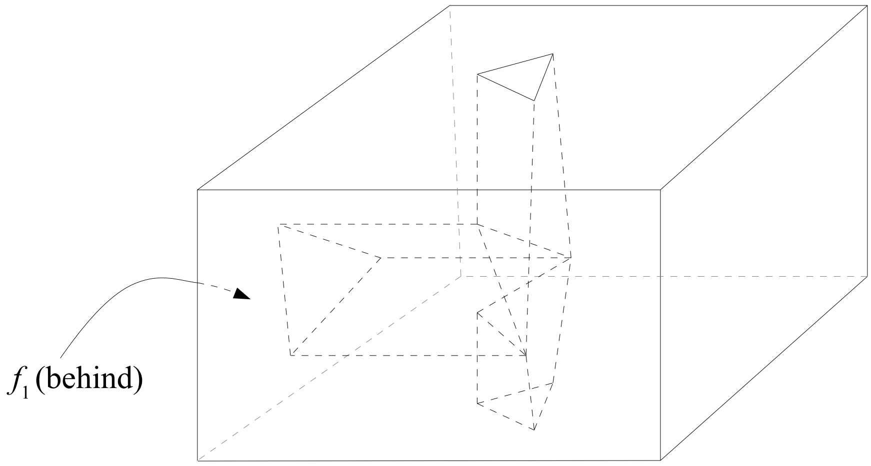


Figure 1: A solid with prismatic holes through it.

a) Give the number of topological entities in this solid (vertices, edges, faces, inner loops (rings), holes and independent solids) and check the Euler-Poincaré relation $\chi(S)=v-e+f-r=2(s-h)$.
b) Describe the face f_1 completely. Your description should include the complete description of edges, co-edges (or boundary edges), vertices, co-vertices (or boundary vertices), loops ... Number them as you wish (but identify them on a sketch of face f_1). Please include the geometrical support for each of the entities. Don't draw links between entities, just refer to them as following (with their names (partial sample, do not copy !):

Boundary Edge BE3	
Edge	E2
Geom. Support	Crv_uv1

Vertex V5	
Geom. Support	Pt_xyz1

Curve Crv_uv1	
Type	Straight line
application	(t) → (u,v)

Edge E2	
Start BV	BV10
End BV	BV3
Geom. Support	Crv_xyz1

Boundary Vertex BV10	
Vertex	V5
Geom. Support	Pt_t1

2) Bézier curves

a) Show that the derivatives of a Bézier curve is a Bézier curve whose control points are the difference between CPs of the original curve.

b) Show formally that the first derivatives of a Bézier curve at its extremities does only depend on the first(or last) few control points.

In the sequel, one wishes to join smoothly two such curves, *P*(*u*), with control points P_i, and *Q*(*u*) with control points Q_i. The junction is made at *u*=1 for *P*, and at *u*=0 for *Q*.

b) What are the conditions for a C^0 continuity and a G^0 continuity on Q_i with respect to the P_i's (fixed)

c) Same for C^1 / G^1 continuity, and C^2 / G^2 continuity.

d) Give De Casteljau's algorithm that is used to compute points on the curve.

e) Show that the vector joining the two points used by the last iteration of De Casteljau's algorithm is proportional to the first derivative of the curve.

Note : $P_i^{d-1}=\sum_{j=0}^{d-1} P_{i+j} B_j^{d-1}(u)$ where P_i^{d-1} are the two points of the penultimate iteration. The response to question a) is needed to answer this question.

General note : Bernstein polynomials are $B_i^d(u)=\binom{d}{i} u^i (1-u)^{d-i}$, with $\binom{d}{i}=\frac{d!}{(d-i)!i!}$. It is easy to show that their derivatives are $B_i^{'d}(u)=d\left(B_{i-1}^{d-1}(u)-B_i^{d-1}(u)\right)$.

(it is not asked to demonstrate it, you can use these results directly if needed)

3) Rational curves and affine invariance

We are now dealing with rational curves expressed in a polar form (as e.g. Bézier), such that :

$$P^w(u)=\sum_0^{N-1} P_i^w \cdot N_i(u)$$

This expression is given in homogeneous coordinates, hence the superscript "*w*" .

a) Give the corresponding expression of the curve *P*(*u*) in 3-dimensional Cartesian coordinates. One may use the matrix notation [...] to refer to individual components of P^w. Please give the expression of *P*(*u*) in a polar form, i.e. as a weighted sum of control points and shape functions $R_i(u)$. Please give the expression of $R_i(u)$.

b) **Demonstrate** that there are conditions on the shape functions $N_i(u)$ to ensure the affine invariance of $P^w(u)$ in the 4D homogeneous space. What are these conditions ?

c) What happens if one is instead interested only on the affine invariance in a more conventional 3D Cartesian space ? Do the conditions on $N_i(u)$ evolve ? If yes, what are these new conditions ? (prove what you say !).

4) Surface modeling and conics

Two curves will be used to generate a surface, one profile curve *P*(*u*), and one trajectory curve *T*(*v*). The idea is to make the profile curve follow the trajectory to define the surface *S*(*u*,*v*). Without loss of generality, the profile curve is contained in the *x*-*z* plane, and the trajectory is in the *x*-*y* plane.

The analytical expression of the surface is the following :

$$S(u,v)=\begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}=\begin{pmatrix} x^p(u)\cdot x^t(v) \\ x^p(u)\cdot y^t(v) \\ z^p(u) \end{pmatrix}, \text{ with } P(u)=\begin{pmatrix} x^p(u) \\ 0 \\ z^p(u) \end{pmatrix} \text{ and } T(v)=\begin{pmatrix} x^t(v) \\ y^t(v) \\ 0 \end{pmatrix}.$$

P(*u*) and *T*(*v*) are in fact rational curves in homogeneous coordinates:

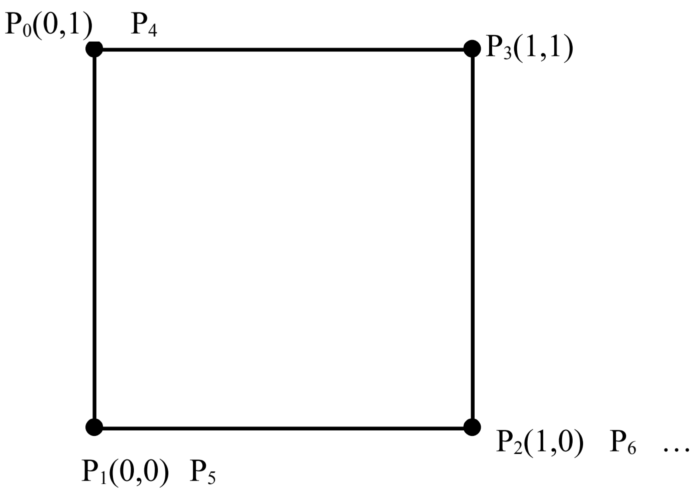
$$P^w(u)=\begin{pmatrix} x^p(u)\cdot w^p(u) \\ 0 \\ z^p(u)\cdot w^p(u) \\ w^p(u) \end{pmatrix} , \quad T^w(v)=\begin{pmatrix} x^t(v)\cdot w^t(v) \\ y^t(v)\cdot w^t(v) \\ 0 \\ w^t(v) \end{pmatrix}$$

a) Give an expression of $S^w(u,v)$ that it is equivalent (under perspective division) to $S(u,v)$, using $P^w(u)$ and $T^w(v)$. (hint : this expression involves only multiplications of components in $P^w(u)$ and $T^w(v)$)
b) $P^w(u)$ and $T^w(v)$ are in fact rational Bézier curves, with control points respectively P_i^w and T_i^w. Give the general expression of $P^w(u)$ and $T^w(v)$ as a function of P_i^w and T_i^w.
c) Give the general expression of $S^w(u,v)$ based on the previous expressions.
d) Now show that $S^w(u,v)$ can be expressed as a weighted sum of shape functions and control points. What are the expressions of the shape functions, and the control points in function of those of $P^w(u)$ and $T^w(v)$?
e) Use the previous results to build a surface of revolution around the z-axis from a given profile curve : what should be the the trajectory curve for a 90 degree turn (please give the control points and the weights) ?
f) Give all the control points of the surface if the control points of the profile curve are the following :

$$P_0^w=\begin{pmatrix}0\\0\\1\\1\end{pmatrix}, P_1^w=\begin{pmatrix}\sqrt{3}/2\\0\\1/2\\1/2\end{pmatrix}, P_2^w=\begin{pmatrix}\sqrt{3}/2\\0\\-1/2\\1\end{pmatrix}.$$

5) Periodic B-Spline curves

We focus on a periodic B-Spline curve of continuity *at least* C_0, for which the shape of the control polygon is given. The nodal sequence is uniform and periodic. The parameter u varies between 0 and 1.



On the picture, a control polygon is given and the control points (CPs) are repeated after one turn, as often as needed for the periodicity requirements.
a) The degree is set to 2. Give a nodal sequence and the exact number of CPs (including CPs that are repeated). Do the curve interpolate any of the points of the control polygon ? How many independent segments do compose the whole curve ? How many CPs do have an influence on each segment ?
b) Same questions if the degree is 3.

6) Generalities

Fill **each** cell in the matrix with *yes* or *no* . A bad answer cancel points, therefore it is advised to respond only if you are sure of your answer – (no answer does not mean "*no*")

	Natural Splines	Cardinal Splines	Bézier curves	B-splines curves	NURBS curves	Bézier triangles	B-Spline surfaces
Allow "periodic" curves / surfaces							
Interpolates the control points							
Has the property of affine invariance							
Allows local control							
Has a polynomial expression (eventually by parts)							
Allows the exact representation of conics							
Allows arbitrary discontinuities in the derivatives							
Allows the exact representation of an "offset" curve / surface*							
Is «variation diminishing»							

* by offset curve , I mean a curve at a constant normal distance from another curve.