1er Master Ingénieur Mécanique & Aérospatiale Monday, January 11th, 2015 – 8:30-12:00 am
CAD Exam Total duration : 3:30 hours.

Course notes, slides and calculators are **NOT** allowed. The wording of all exercises should be returned to the instructor.

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1) Generalities on parametric surfaces.

Let $S(u,v) = (x(u,v), y(u,v), z(u,v))^T$ be a parametric surface, and $C_{uv}(t) = (u(t), v(t))^T$ a parametric curve defined in the parametric space of $S(u,v)$.

a- Let $C_{xyz}(t)$ be the expression in global coordinates (Oxyz) of C_{uv}, based on the expression of surface S.
What are the conditions of regularity of the curve $C_{uv}(t)$, $C_{xyz}(t)$, and the surface $S(u,v)$? Does the regularity of C_{uv} and S always imply the regularity of $C_{xyz}(t)$?
b- Give the expression of the length of any curve C in 3D, between $t_{beg}=t_1$ and $t_{end}=t_2$.
c- Using the expression of C_{xyz}, based on C_{uv} and S, express the length l of C_{xyz} using only partial derivatives of C_{uv} (with respect to t) and of S (with respect to u and v).
d- Rearrange terms so that the matrix corresponding to the first fundamental form $M_1 = \begin{pmatrix} e & f \\ f & g \end{pmatrix}$, appears in the expression and give the expression of e, f and g.
e- Give the expression of the area of a patch on surface S, based on partial derivatives of S. (The limits of the patch are not important here). Transform the notation so that it uses the matrix M_1.

2) B-Splines curves.

One wishes to represent a closed curve that has multiple symmetries. On the figure below; the black original pattern is rotated three times to make the complete closed curve.

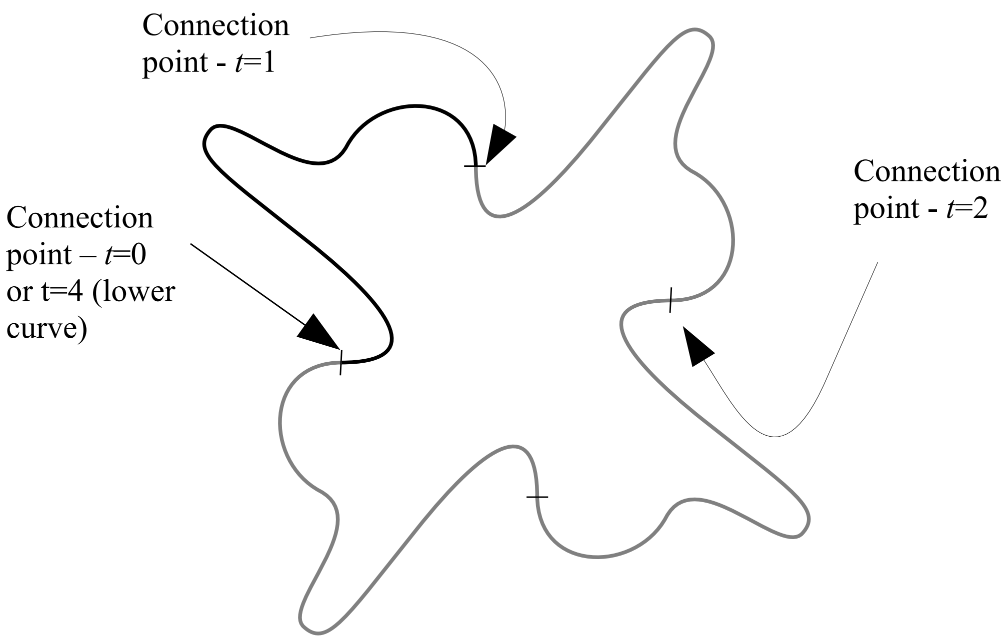

Figure 1- periodic curve

a- The desired continuity at the connection points is C_2. What is the minimal degree of the B-Spline curve that is required to achieve such a continuity ? We now set the degree to 2, what is the maximum continuity that one can achieve, regardless of the position of the control points ?

In the following questions, degree *d*=2 is used.

b- One wishes to represent the 1st parts of the curve (in black), with a conventional B-Spline interpolating the 1st and last control points. How many control points are possibly needed ? Place them approximately on the following drawing, and give the related nodal sequence. How many independent polynomial arcs compose this part ? Identify them on the sketch. Please take notice that the number of wiggles in the curve is linked to the number of wiggles in the control polygon ...

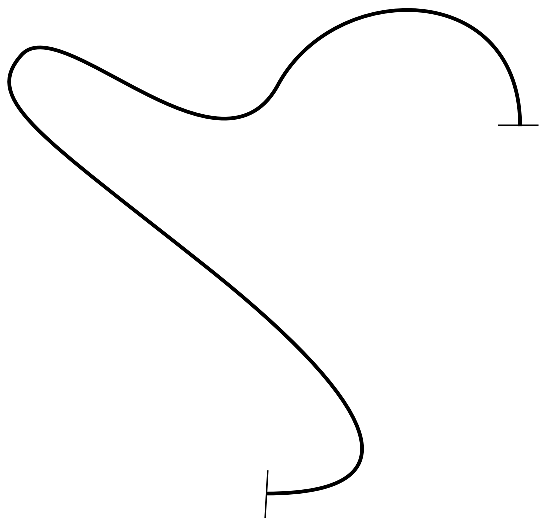

Figure 1-b Original arc.

c- We now wish to represent the same arc with a uniform nodal sequence. Give such a nodal sequence that allows to compute the whole arc with the same interval for the parameter as in the previous question, and the same number of independent polynomial intervals as in question b. Is it possible to represent exactly the same curve, although the nodal sequence is different ? Why ?
d- Now, the endpoints are not interpolated anymore, therefore one has to compute the right position for them. The recurrence formula of Cox-de Boor for the computation of B-Spline shape functions is the following :

if $d=0$: $N_i^0(u)=\begin{cases} 1 & \text{if} \quad u_i \le u < u_{i+1} \\ 0 & \text{otherwise} \end{cases}$

otherwise $N_i^d(u)=\dfrac{u-u_i}{u_{i+d}-u_i} N_i^{d-1}(u)+\dfrac{u_{i+d+1}-u}{u_{i+d+1}-u_{i+1}} N_{i+1}^{d-1}(u)$

Using these relations, determine, for a degree 2 curve having a uniform nodal sequence, the position of the extremities of the curve with respect to the control points. What about the tangents at the extremities ?
e- Give the nodal sequence of the whole periodic curve. How many control points ? How many of them must be duplicated so that it is really periodic ? Place the control points approximately on figure 1 and identify those that must be repeated. In this case, each arch must match the description you choose in question c, for the sake of coherence.

3) **NURBS curves and surfaces, and conic sections.**

One would like to represent a patch has a complex revolution shape with a NURBS surface. For this, one has to determine the position of the control points of a circular arc and compute their weights. As shown in the following figure, the patch encompasses an angle of 45 along C.

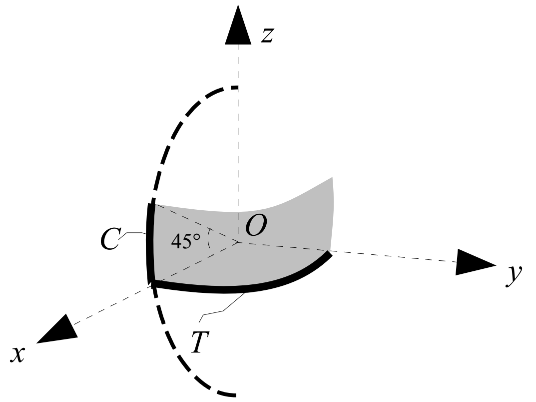

Figure 2: generalized revolution patch.

a- What does the acronym "NURBS" stand for ? Explain briefly the meaning of the terms.
b- The control points in homogeneous coordinates and the nodal sequence of the curve T are the following :

$$T_0^w = \begin{pmatrix}1\\0\\0\\1\end{pmatrix} \quad T_1^w = \begin{pmatrix}2\\1\\0\\2\end{pmatrix} \quad T_2^w = \begin{pmatrix}1\\2\\0\\2\end{pmatrix} \quad T_3^w = \begin{pmatrix}0\\1\\0\\1\end{pmatrix}, \quad V = [0,0,0,1/2,1,1,1]$$

Compute the actual positions **in 3D** of these control points, and sketch them on figure 2.
c- Give the degree, the nodal sequence, the number of control points to represent a 45-degrees circular arc.
d- Compute the position of the control points to represent the arc *C* in figure 2. 3D positions and weights (or homogeneous coordinates) **and the proof** leading to these values are asked (you can help yourself with a sketch).
e- Now imagine one have a profile curve (*C*(*u*) in our case) , with its nodal sequence U, control points P_i and weights, and a trajectory curve (e.g. *T*(*v*)), also with its own nodal sequence V, control points T_j and weights, construct a generalized surface of revolution by combining the control points of both curves such that one obtains the followin surface (analytic expression) :

With $C(u) = \begin{pmatrix} x^c(u) \\ 0 \\ z^c(u) \end{pmatrix}$ and $T(v) = \begin{pmatrix} x^t(v) \\ y^t(v) \\ 0 \end{pmatrix}$ one gets $S(u,v) = \begin{pmatrix} x^c(u)\cdot x^t(v) \\ x^c(u)\cdot y^t(v) \\ z^c(u) \end{pmatrix}$.

Hint : Start with the original definitions of *C* and *T* as NURBS curves :

$$C^w(u) = \sum_{i=0}^{n} N_i^p(u) P_i^w \quad \text{and} \quad T^w(v) = \sum_{j=0}^{m} N_j^q(v) T_j^w .$$

e- From the results of question d, build the patch (in grey on figure 2). Give the final control points (3D position and weights), assuming that the trajectory curve has the control points and nodal sequence given in question b, and the profile curve is the circular arc of question d.

4) Subdivision scheme and Bézier curves.

One wish to draw a curve by using a specific subdivision scheme. The scheme is based on the vertices of a polyline having an even number of sides. From $\{P_0, P_1, \dots P_{2n}\}$, compute :

$$E_i = \frac{1}{2} P_i + \frac{1}{2} P_{i+1}$$

$$B_{2i} = P_{2i}$$

$$V_{2i+1} = \frac{1}{2} E_{2i} + \frac{1}{2} E_{2i+1}$$

, then use $\{B_0, E_0, V_1, E_1, \dots, B_{2i}, E_{2i}, V_{2i+1}, E_{2i+1}, \dots B_{2n-2}, E_{2n-2}, V_{2n-1}, E_{2n-1}, B_{2n}\}$ as new vertices for the polyline at next step $\{P'_0, P'_1, \dots P'_{4n}\}$.

a- Show that this scheme has the property of affine invariance.
b- Lets start with a polyline made of only three vertices P_0 - P_1 - P_2 (see figure). Show that, when one repeats the scheme an infinite number of times, the limiting curve is a parametric curve having the following definition that is based on the same original vertices:

$$P(u) = P_0 \cdot (1-u)^2 + P_1 \cdot 2u(1-u) + P_2 \cdot u^2 = [1 \;\; u \;\; u^2] \cdot \begin{bmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} P_0 \\ P_1 \\ P_2 \end{bmatrix}$$

c- Show that the above expression corresponds to that of a Bezier curve of degree 2, using e.g. De Casteljau's algorithm.

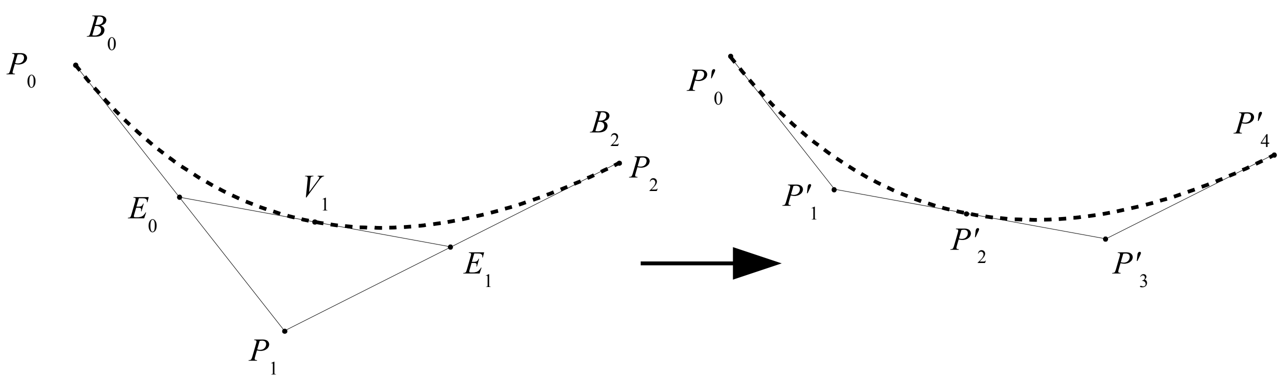

Figure 3: first step of the subdivision scheme on a minimal polyline (the limit curve is dashed).

5) Splines

Sometimes, mechanical parts are manufactured directly as copies from samples given by a customer. Obviously, these samples must be digitized so that e.g. CNC milling machines could be used and price be reduced. The digitizer gives a file containing 2D points; and for each point, an orientation (tangent of the surface):

```
ID   x        y        dx        dy
0    12.34    -45.67   0.45567   -0.89015
1    13.12    -40.00   0.33345   -0.94277
...
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The list of points have the following properties
- it is an oriented list (because of the tangent vector and the ordering)
- the curve is open at the extremities (not closed)
- points are never repeated, and the density of points is depending on the curvature of the sample, yielding a roughly constant geometrical error (chord error)
- the sampling error for each point is practically negligible
- the tangent is a unit vector.

Now, one wishes to build an interpolation of the curve that has a G^1 continuity, using cubic splines.

a) What are the equations that must be satisfied for each spline segment between two consecutive interpolation points (do as if the derivatives were known at the segment's extremities) ? You will choose a unit parametrization [0...1] for each interval $[P_i\text{-}P_{i+1}]$.

b) The tangent that is given in the file is normalized (its norm is always equal to one). These tangents are used directly to compute the spline. Now, if the coordinates of the points are scaled (multiplied by a constant), how will the shape of the curve evolve ? Prove that affine invariance is indeed not satisfied.

c) The previous question shows that using the normalized tangents directly in the equations is not feasible. Propose another way of using the tangents so that affine invariance is satisfied. Show that your definition also enforces a C^1 continuity (not only G^1).

6) Generalities

Fill **each** cell in the matrix with *yes* or *no* . A bad answer cancel points, therefore it is advised to respond only if you are sure of your answer – (no answer does **not** mean "*no*" - it means you don't know !)

	Coons patch	B-spline curve/surf.	Subdivision surface	Bézier curve/surf.	NURBS curve/suf.	Natural Spline
Allow "periodic" curves / surfaces						
Interpolates the control points						
Has the property of affine invariance						
Allows the exact representation of conics						
Has always a polynomial expression (also by parts)						
Allows local control						
"Offset" curve / surface* exactly represented						
Allows arbitrary discontinuities in the derivatives						
Unique analytic representation (not by parts)						
Is «variation diminishing»						

* by offset curve/surface , it is meant a curve/surface at a constant normal distance from another curve/surface.