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On the curvature of curves and surfaces defined by normalforms

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Abstract

The normalform $h = 0$ of a curve (surface) is a generalization of the Hesse normalform of a line in $\mathbb{R}^2$ (plane in $\mathbb{R}^3$). It was introduced and applied to curve and surface design in recent papers. For determining the curvature of a curve (surface) defined via normalforms it is necessary to have formulas for the second derivatives of the normalform function h depending on the unit normal and the normal curvatures of three tangential directions of the surface. These are derived and applied to visualization of the curvature of bisectors and blending curves, isophotes, curvature lines, feature lines and intersection curves of surfaces. The idea of the normalform is an appropriate tool for proving theoretical statements, too. As an example a simple proof of the Linkage Curve Theorem is given.

Keywords: Normalform; Hessian matrix; Curvature; Normal curvature; Bisector; G^n-blending; G^2-continuity; Umbilic points; Isophote; Curvature line; Feature line; Ridge; Ravine; Intersection curve; Foot point

1. Introduction

The normalform $h = 0$ of a curve (surface) is a generalization of the Hesse normalform of a line in $\mathbb{R}^2$ (plane in $\mathbb{R}^3$). It was introduced and applied to curve (surface) design in (Hartmann, 1998a, 1998b). But only in rare cases (lines and circles in $\mathbb{R}^2$, planes and spheres in $\mathbb{R}^3$) the normalform function h is known explicitly. So the evaluation of h for a point $\mathbf{x}$ is done usually by determining the corresponding foot point $\mathbf{x}_0$ on the curve (surface). Then $h(\mathbf{x})$ is just the suitably oriented distance $\|\mathbf{x} - \mathbf{x}_0\|$ and the gradient $\nabla h(\mathbf{x})$ is the unit normal at the foot point. The essential advantage in using normalforms is that nearly all curves (surfaces) can be treated uniformly as implicit curves (surfaces).

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So simple algebraic manipulations solve difficult blending and approximation problems (cf. (Hartmann, 1998a, 1998b)). The result of such a manipulation is in any case an implicit curve (surface) only depending on the geometry of the involved curves (surfaces) and not on the used representation (parametric, implicit,...) which is accidentally chosen. For visualization or intersection of implicit surfaces it is sufficient to be able to calculate $h(\mathbf{x})$ and $\nabla h(\mathbf{x})$. Concerning any question about the curvature and its visualization one needs the second derivatives (Hessian matrix H_h) of the normalform function h. The Hessian matrix $H_h(\mathbf{x})$ at a point $\mathbf{x}$ depends on the curvature (normal curvatures) at the corresponding foot point $\mathbf{x}_0$. The main result of this paper are formulas for the Hessian matrix H_h depending on the unit normal and the curvature (three normal curvatures) of arbitrarily defined curves (surfaces), especially for implicitly and parametrically defined curves (surfaces). Hence the curvature(s) of a curve (surface) which is defined via normalforms of other curves (surfaces) can be calculated and visualized (cf. Section 5).

In Section 2 the normalform of a curve/surface is defined and a fundamental property of the normalform function h is given. Section 3 deals with the case of a planar curve. The gradient ∇h and the Hessian matrix H_h of the normalform function h are derived as functions of geometric properties. The examples of implicit curves and parametric curves are considered in detail. Section 4 contains the formulas for surfaces with special attention to implicit and parametric surfaces. Furthermore, the normalform is applied to prove the Linkage Curve Theorem. In Section 5 we introduce stable foot point algorithms for curves and surfaces and use the derived results for visualizing the curvature of blending curves and surfaces.

2. The normalform of a curve/surface

Analogously to the Hesse normalform of a line in $\mathbb{R}^2$ and a plane in $\mathbb{R}^3$ we define the normalform for a curve and a surface.

Definition. Let $\Gamma(\Phi)$ be a smooth implicit curve (surface) $h = 0$ in $\mathbb{R}^2(\mathbb{R}^3)$. If the function h is continuously differentiable and $\|\nabla h\| = 1$ on $\Gamma(\Phi)$ and in a vicinity of $\Gamma(\Phi)$ then the equation $h = 0$ is called *normalform* of $\Gamma(\Phi)$ and h the corresponding *normalform function* or (as of its geometrical meaning) *oriented distance function*.

We get the following result out of the theory of the nonlinear partial differential equation $\|\nabla h\|^2 = 1$ (cf. Courant and Hilbert, 1962, p. 88; Weise, 1966, p. 193; Gilbarg and Trudinger, 1983, p. 355):

Result. *Let $\Gamma(\Phi)$ be a C^2-continuous curve (surface) in $\mathbb{R}^2(\mathbb{R}^3)$. Then there exists in a vicinity of $\Gamma(\Phi)$ a unique differentiable function h such that $h = 0$ is the normalform of $\Gamma(\Phi)$.*

If $\Gamma(\Phi)$ is of continuity class C^n then function h, too.

h fulfills the fundamental equation

$$\boxed{h\big(\mathbf{x} + \delta\nabla h(\mathbf{x})\big) = h(\mathbf{x}) + \delta.}$$

If $\mathbf{x} \in \Gamma(\Phi)$ then $h(\mathbf{x} + \delta\nabla h(\mathbf{x})) = \delta$ is the (oriented) distance of point $\mathbf{x} + \delta\nabla h(\mathbf{x})$ to the curve (surface) and $\mathbf{x}$ is the foot point of $\mathbf{x} + \delta\nabla h(\mathbf{x})$ on the curve (surface). Equation $h = \delta$ describes the offset curve (surface) of distance δ. Hence offset curves (surfaces) have the same geometric continuity as the base curve (surface) $h = 0$ (This is Theorem 1 of (Hermann, 1998)).

Remark. (a) The existence and the continuity statement of the above result can also be obtained from Section 3.1 in (Hartmann, 1998a).

(b) The continuity assumption C^n, $n \geqslant 2$, can be reduced to C^1-continuous curves and surfaces which are piecewise C^2. A further reduction of the precondition is contained in (Ostrowski, 1956).

Function h is known explicitly in rare cases only. In general the evaluation of h is done numerically by determining foot points (cf. Section 5). Once the foot point of a point in the vicinity is known, the first and second derivatives of h can be evaluated using the normal vector and the curvature(s) of the curve (surface) (see below).

3. The first and second derivatives of the normalform of a planar curve

3.1. *The first and second derivatives on the curve*

Let $h(x, y) = 0$ be the normalform of a planar curve Γ_0 with continuous first and second derivatives. Hence, $h_x^2 + h_y^2 = 1$. Differentiating this equation yields

$$h_x h_{xx} + h_y h_{xy} = 0,$$
$$h_x h_{xy} + h_y h_{yy} = 0.$$

Properties of the Hessian matrix H_h of function h are:

- ∇h is eigenvector of H_h with eigenvalue $\lambda_1 = 0$.
  There exists a second eigenvalue $\lambda_2 = \kappa$ with the tangent vector $\mathbf{t} := (-h_y, h_x)$ as eigenvector. $(\nabla h)'\mathbf{t}^{\mathrm{T}} = H_h\mathbf{t}^{\mathrm{T}} = \kappa\mathbf{t}$ describes the change of the unit normal vector ∇h in tangent direction and $\mathbf{t}H_h\mathbf{t}^{\mathrm{T}} = \kappa$ its amount. Hence κ is the *curvature* of curve $h = 0$.

  $$\kappa = h_y^2 h_{xx} - 2h_x h_y h_{xy} + h_x^2 h_{yy}.$$

- The *characteristic polynomial* of H_h is $\lambda^2 - (h_{xx} + h_{yy})\lambda$ and

  $$\det(H_h) = 0, \quad \kappa = h_{xx} + h_{yy} \quad \text{and} \quad H_h^2 = \kappa H_h.$$

The determinant of the linear system

$$\begin{aligned} h_x h_{xx} \qquad\qquad\qquad &+ h_y h_{xy} = 0, \\ h_y h_{yy} \quad &+ h_x h_{xy} = 0, \\ h_y^2 h_{xx} + h_x^2 h_{yy} &- 2h_x h_y h_{xy} = \kappa \end{aligned}$$

for h_{xx}, h_{yy}, h_{xy} is -1. Hence, the Hessian matrix H_h of the normalform function h can be expressed at curve point (x, y) by the gradient $\nabla h(x, y)$ (unit normal) and the curvature $\kappa(x, y)$:

$$\boxed{H_h := \begin{pmatrix} h_{xx} & h_{xy} \\ h_{yx} & h_{yy} \end{pmatrix} = \kappa \begin{pmatrix} h_y^2 & -h_x h_y \\ -h_x h_y & h_x^2 \end{pmatrix} = \kappa(-h_y, h_x)^{\mathrm{T}}(-h_y, h_x).}$$

3.1.1. *Example 1: Implicit curve*

Let Γ: $f(x, y) = 0$ be a regular implicit curve with continuous second derivatives of function f and h its normalform function.

For a curve point (x, y) we have

(1) $h = 0$,

(2) $\nabla h = \dfrac{\nabla f}{\|\nabla f\|}$,

(3)
$$\kappa = \frac{(-f_y, f_x)}{\|\nabla f\|}\left(\frac{\nabla f}{\|\nabla f\|}\right)'\left(\frac{(-f_y, f_x)}{\|\nabla f\|}\right)^{\mathrm{T}}$$
$$= \frac{(-f_y, f_x)}{\|\nabla f\|}\left(\frac{H_f}{\|\nabla f\|} - \cdots \nabla f\right)\left(\frac{(-f_y, f_x)}{\|\nabla f\|}\right)^{\mathrm{T}}$$
$$= \frac{f_y^2 f_{xx} - 2 f_x f_y f_{xy} + f_x^2 f_{yy}}{(f_x^2 + f_y^2)^{3/2}} \text{ (amount of change of unit normal vector),}$$

$$H_h = \kappa \begin{pmatrix} h_y^2 & -h_x h_y \\ -h_x h_y & h_x^2 \end{pmatrix} = \frac{1}{\|\nabla f\|^5} \begin{vmatrix} f_{xx} & f_{xy} & f_x \\ f_{yx} & f_{yy} & f_y \\ f_x & f_y & 0 \end{vmatrix} \begin{pmatrix} f_y^2 & -f_x f_y \\ -f_x f_y & f_x^2 \end{pmatrix}.$$

3.1.2. *Example 2: Parametric curve*

Let Γ_0: $\mathbf{x} = \mathbf{c}(t) = (c_1(t), c_2(t))$ be a regular parametrically defined curve with continuous second derivatives and h its normalform function.

For a curve point $\mathbf{c} = (c_1, c_2)$ we have

(1) $h(\mathbf{c}) = 0$,

(2) $\nabla h(\mathbf{c}) = (-\dot{c}_2, \dot{c}_1)/\|\dot{\mathbf{c}}\|$,

(3) $\kappa(\mathbf{c}) = \det(\ddot{\mathbf{c}}, \dot{\mathbf{c}})/\|\dot{\mathbf{c}}\|^3$,

$$H_h(\mathbf{c}) = \kappa \begin{pmatrix} h_y^2 & -h_x h_y \\ -h_x h_y & h_x^2 \end{pmatrix} = \frac{\det(\ddot{\mathbf{c}}, \dot{\mathbf{c}})}{\|\dot{\mathbf{c}}\|^5} \begin{pmatrix} \dot{c}_1^2 & \dot{c}_1\dot{c}_2 \\ \dot{c}_1\dot{c}_2 & \dot{c}_2^2 \end{pmatrix}.$$

3.2. The first and second derivatives in the vicinity of the curve

Let $\mathbf{x} \in \Gamma_0$ be in such a way that h is C^2-continuous with curvature κ. The distance parameter $\delta \in \mathbb{R}$ is chosen so that $1 + \delta\kappa(\mathbf{x}) > 0$ and h is C^2-continuous at point $\mathbf{x}_\delta := \mathbf{x} + \delta\nabla h(\mathbf{x})$. Differentiating the fundamental equation $h(\mathbf{x}_\delta) = h(\mathbf{x}) + \delta$ yields:

$$\big(I + \delta H_h(\mathbf{x})\big)\nabla h(\mathbf{x}_\delta) = \nabla h(\mathbf{x}),$$

with Hessian matrix H_h and 2×2 unit matrix I. This is a linear system for vector $\nabla h(\mathbf{x}_\delta)$ with determinant $\det(I + \delta H_h(\mathbf{x})) = 1 + \delta\kappa \neq 0$. The unique solution is

$$\boxed{\nabla h(\mathbf{x}_\delta) = \nabla h(\mathbf{x}).}$$

(For the proof remember that $\nabla h(\mathbf{x})$ is eigenvector of H_h with eigenvalue 0.)

Differentiating this equation yields

$$\big(I + \delta H_h(\mathbf{x})\big)H_h(\mathbf{x}_\delta) = H_h(\mathbf{x}).$$

The unique solution of this linear system for $H_h(\mathbf{x}_\delta)$ is

$$H_h(\mathbf{x}_\delta) = \frac{H_h(\mathbf{x})}{1 + \delta\kappa(\mathbf{x})} = \frac{\kappa(\mathbf{x})}{1 + \delta\kappa(\mathbf{x})}\begin{pmatrix} h_y^2(\mathbf{x}) & -h_x(\mathbf{x})h_y(\mathbf{x}) \\ -h_x(\mathbf{x})h_y(\mathbf{x}) & h_x^2(\mathbf{x}) \end{pmatrix}.$$

(For the proof use the identity $H_h^2 = \kappa H_h$.)

The curvature of the offset curve $\Gamma_\delta : h = \delta$ (of Γ_0) at point $\mathbf{x}_\delta$ is

$$\boxed{\kappa(\mathbf{x}_\delta) = \frac{\kappa(\mathbf{x})}{1 + \delta\kappa(\mathbf{x})}.}$$

($\mathbf{x}$ is the foot point of $\mathbf{x}_\delta$ on Γ_0.)

4. The second derivatives of the normalform of a surface

4.1. The second derivatives on the surface

Let $h(x, y, z) = 0$ be the normalform of a surface Φ_0 with continuous first and second derivatives of h. Differentiating $h_x^2 + h_y^2 + h_z^2 = 1$ yields

$$h_x h_{xx} + h_y h_{xy} + h_z h_{xz} = 0, \tag{1}$$

$$h_x h_{xy} + h_y h_{yy} + h_z h_{yz} = 0, \tag{2}$$

$$h_x h_{xz} + h_y h_{yz} + h_z h_{zz} = 0. \tag{3}$$

We get the following properties and applications of the Hessian matrix of the normalform function h:

- Considerations analogous to the plane case show:
  The *normal curvature* for unit tangent direction $\mathbf{v}$ is $\kappa_n = \mathbf{v}H_h\mathbf{v}^{\mathrm{T}}$.
  (κ_n is the curvature of the surface curve contained in the normal plane determined by point $\mathbf{x}$, the gradient ∇h and the tangent vector $\mathbf{v}$.)

- The *eigenvalues* of the Hessian matrix H_h are
  $\lambda_1 = 0$ with eigenvector ∇h,
  $\lambda_2 = \kappa_{\min}$, $\lambda_3 = \kappa_{\max}$ (main curvatures).
- The *characteristic polynomial* of H_h is $-\lambda^3 + 2H\lambda^2 - K\lambda$ with
  mean curvature
  $H := \frac{1}{2}(\kappa_{\min} + \kappa_{\max}) = \frac{1}{2}(h_{xx} + h_{yy} + h_{zz})$ and
  Gaussian curvature
  $K := \kappa_{\min}\kappa_{\max} = \begin{vmatrix} h_{xx} & h_{xy} \\ h_{yx} & h_{yy} \end{vmatrix} + \begin{vmatrix} h_{xx} & h_{xz} \\ h_{zx} & h_{zz} \end{vmatrix} + \begin{vmatrix} h_{yy} & h_{yz} \\ h_{zy} & h_{zz} \end{vmatrix}.$
  The *minimal polynomial* is the characteristic polynomial if $\kappa_{\min} \neq \kappa_{\max}$ (*nonumbilic case*) and $\lambda^2 - \kappa\lambda$ if $\kappa := \kappa_{\min} = \kappa_{\max}$ (*umbilic case*).
  (Prove it after diagonalizing H_h!)
- There exist two orthogonal unit eigenvectors $\mathbf{v}_{\min}$, $\mathbf{v}_{\max}$ corresponding to $\kappa_{\min}$ and $\kappa_{\max}$ respectively. Hence the curvature for direction $\mathbf{v}(\varphi) := \mathbf{v}_{\min}\cos\varphi + \mathbf{v}_{\max}\sin\varphi$ is

  $$\kappa(\varphi) = \kappa_{\min}\cos^2\varphi + \kappa_{\max}\sin^2\varphi,$$

  which is the well known *Euler formula*.
  For $\kappa_{\min} \neq \kappa_{\max}$ (nonumbilic case) $\mathbf{v}_{\min}, \mathbf{v}_{\max}$ are called *principal directions*. For determining the principal directions we introduce local base vectors in the tangent plane:
  $\mathbf{e}_1 := (h_y, -h_x, 0)/\|\cdots\|$ if ($h_x > 0.5$ or $h_y > 0.5$) else
  $\mathbf{e}_1 := (-h_z, 0, h_x)/\|\cdots\|$ and $\mathbf{e}_2 := \nabla h \times \mathbf{e}_1$.
  A unit eigenvector $\mathbf{v}_{\min}$ belonging to main curvature $\kappa_{\min}$ can be written as $\mathbf{v}_{\min} = \xi\mathbf{e}_1 + \eta\mathbf{e}_2$. Inserting this equation into the linear system $(H_h - \kappa_{\min}I)\mathbf{x}^{\mathrm{T}} = \mathbf{0}^{\mathrm{T}}$ yields: ξ, η are solutions of the system

  $$\alpha\xi + \beta\eta = 0, \quad \xi^2 + \eta^2 = 1,$$

  with $\alpha := \mathbf{e}_1(H_h - \kappa_{\min}I)\mathbf{e}_1^{\mathrm{T}} = \mathbf{e}_1 H_h \mathbf{e}_1^{\mathrm{T}} - \kappa_{\min}$ and if $\alpha \neq 0$ then $\beta := \mathbf{e}_1(H_h - \kappa_{\min}I)\mathbf{e}_2^{\mathrm{T}} = \mathbf{e}_1 H_h \mathbf{e}_2^{\mathrm{T}}$ else $\beta := 1$.
  Hence

  $$\mathbf{v}_{\min} = \frac{\beta\mathbf{e}_1 - \alpha\mathbf{e}_2}{\sqrt{\alpha^2 + \beta^2}}, \qquad \mathbf{v}_{\max} = \frac{\alpha\mathbf{e}_1 + \beta\mathbf{e}_2}{\sqrt{\alpha^2 + \beta^2}}.$$

  Numerical instabilities ($\alpha \approx 0, \beta \approx 0$) may occur if $\kappa_{\min} \approx \kappa_{\max}$ or $\mathbf{e}_1 \approx \mathbf{v}_{\min}$. The second reason can be omitted by rotating the bases $\mathbf{e}_1, \mathbf{e}_2$ by 45°. Further discussions of numerical instabilities for parametrically defined surfaces are contained in (Farouki, 1998).

Remark. In order to respect the first three equations, as mentioned above, we rewrite the elements of the Hessian in an asymmetrical way:

$$\begin{aligned} h_{xx} &= \left(h_x^2 + h_y^2 + h_y^2\right)h_{xx} = \left(h_y^2 + h_z^2\right)h_{xx} + h_x^2 h_{xx} \\ &= \left(h_y^2 + h_z^2\right)h_{xx} - h_x h_y h_{xy} - h_x h_z h_{xz} \quad \text{(Eq. (1) used)}, \\ h_{xy} &= \left(h_x^2 + h_z^2\right)h_{xy} - h_x h_y h_{xx} - h_x h_z h_{xz} \quad \text{(Eq. (1) used)}, \end{aligned}$$

$$h_{yx} = \big(h_y^2 + h_z^2\big)h_{xy} - h_x h_y h_{yy} - h_x h_z h_{yz} \quad \text{(Eq. (2) used)},$$
$$\ldots$$

By using the equation $h_x^2 + h_y^2 + h_z^2 = 1$ and the formula for the mean curvature H and Gaussian curvature K respectively one gets

$$-2H = \begin{vmatrix} h_{xx} & h_{xy} & h_x \\ h_{yx} & h_{yy} & h_y \\ h_x & h_y & 0 \end{vmatrix} + \begin{vmatrix} h_{yy} & h_{yz} & h_y \\ h_{yz} & h_{zz} & h_z \\ h_y & h_z & 0 \end{vmatrix} + \begin{vmatrix} h_{xx} & h_{xz} & h_x \\ h_{xz} & h_{zz} & h_z \\ h_x & h_z & 0 \end{vmatrix},$$

$$-K = \begin{vmatrix} h_{xx} & h_{xy} & h_{xz} & h_x \\ h_{yx} & h_{yy} & h_{yz} & h_y \\ h_{zx} & h_{zy} & h_{zz} & h_z \\ h_x & h_y & h_z & 0 \end{vmatrix}.$$

4.1.1. Case $h_x h_y h_z \neq 0$

In order to get further three linear conditions for the six second derivatives of h (on the surface) we consider the normal curvatures $\kappa_i = \mathbf{v}_i H_h \mathbf{v}_i^{\mathrm{T}}$ for the three tangent vectors

$$\mathbf{v}_1 = \frac{(0, h_z, -h_y)}{\sqrt{h_y^2 + h_z^2}}, \qquad \mathbf{v}_2 = \frac{(h_z, 0, -h_x)}{\sqrt{h_x^2 + h_z^2}}, \qquad \mathbf{v}_3 = \frac{(h_y, -h_x, 0)}{\sqrt{h_x^2 + h_y^2}}.$$

Hence the second derivatives of h fulfill the linear system:

$$\begin{aligned}
h_x h_{xx} \quad\quad\quad\quad\quad & + h_y h_{xy} \quad\quad\quad\quad + h_z h_{xz} &= 0,\\
h_y h_{yy} \quad\quad\quad & + h_x h_{xy} + h_z h_{yz} &= 0,\\
h_y h_{zz} \quad\quad & + h_y h_{yz} + h_x h_{xz} &= 0,\\
h_y^2 h_{xx} + h_x^2 h_{yy} \quad & - 2h_x h_y h_{xy} &= \kappa_3\big(h_x^2 + h_y^2\big),\\
h_z^2 h_{yy} + h_y^2 h_{zz} \quad & - 2h_y h_z h_{yz} &= \kappa_1\big(h_y^2 + h_z^2\big),\\
h_z^2 h_{xx} + h_x^2 h_{zz} \quad & - 2h_x h_z h_{xz} &= \kappa_2\big(h_x^2 + h_z^2\big).
\end{aligned}$$

With respect to the identity $h_x^2 + h_y^2 + h_z^2 = 1$ the determinant is $-2h_x h_y h_z$. Applying CRAMER's rule and using the annotations

$$k_1 := \kappa_1\big(h_y^2 + h_z^2\big), \quad k_2 := \kappa_2\big(h_x^2 + h_z^2\big), \quad k_3 := \kappa_3\big(h_x^2 + h_y^2\big)$$

and the identity $h_x^2 + h_y^2 + h_z^2 = 1$ yield

$$\begin{aligned}
h_{xx} &= -(k_1 + k_2 + k_3)h_x^2 + k_2 + k_3,\\
h_{yy} &= -(k_1 + k_2 + k_3)h_y^2 + k_3 + k_1,\\
h_{zz} &= -(k_1 + k_2 + k_3)h_z^2 + k_1 + k_2,
\end{aligned}$$

$$h_{xy} = -(k_1 + k_2 + k_3)h_x h_y + \frac{k_1 h_x^2 + k_2 h_y^2 - k_3 h_z^2}{2h_x h_y},$$
$$h_{yz} = -(k_1 + k_2 + k_3)h_y h_z + \frac{-k_1 h_x^2 + k_2 h_y^2 + k_3 h_z^2}{2h_y h_z},$$
$$h_{xz} = -(k_1 + k_2 + k_3)h_x h_z + \frac{k_1 h_x^2 - k_2 h_y^2 + k_3 h_z^2}{2h_x h_z}.$$

Remark. (a)$-\frac{1}{2}(k_1 + k_2 + k_3)$ is the mean curvature of the surface at the surface point of consideration.

(b) The Hessian matrix of the normalform function h is uniquely determined by the unit normal (h_x, h_y, h_z) and the three curvatures κ_1, κ_2, κ_3.

4.1.2. Case $h_z = 0$

Now we assume that $h_z = 0$. Hence, $h_x^2 + h_y^2 = 1$.

We choose the following three special tangent vectors:

$$\mathbf{v}_1 = \frac{(h_y, -h_x, 1)}{\sqrt{2}}, \qquad \mathbf{v}_2 = \frac{(-h_y, h_x, 1)}{\sqrt{2}}, \qquad \mathbf{v}_3 = (0, 0, 1).$$

Hence the six second derivatives of h fulfill the linear system:

$$\begin{aligned}
&h_x h_{xx} && && &&+h_y h_{xy} && &&+h_z h_{xz} &&= 0,\\
& &&h_y h_{yy} && &&+h_x h_{xy} &&+h_z h_{yz} && &&= 0,\\
& && &&h_y h_{zz} && &&+h_y h_{yz} &&+h_x h_{xz} &&= 0,\\
&h_y^2 h_{xx} &&+h_x^2 h_{yy} &&+h_{zz} &&-2h_x h_y h_{xy} &&-2h_x h_{yz} &&+2h_y h_{xz} &&= 2\kappa_1,\\
&h_y^2 h_{xx} &&+h_x^2 h_{yy} &&+h_{zz} &&-2h_x h_y h_{xy} &&+2h_x h_{yz} &&-2h_y h_{xz} &&= 2\kappa_2,\\
& && &&h_{zz} && && && &&= \kappa_3.
\end{aligned}$$

The determinant is 4. Applying CRAMER's rule and the identity $h_x^2 + h_y^2 = 1$ yield

$$\boxed{\begin{aligned}
&h_{xx} = h_y^2(\kappa_1 + \kappa_2 - \kappa_3), \quad h_{yy} = h_x^2(\kappa_1 + \kappa_2 - \kappa_3), \quad h_{zz} = \kappa_3,\\
&h_{xy} = -h_x h_y(\kappa_1 + \kappa_2 - \kappa_3), \quad h_{yz} = h_x(\kappa_2 - \kappa_1)/2, \quad h_{xz} = h_y(\kappa_1 - \kappa_2)/2.
\end{aligned}}$$

The cases $h_x = 0$ and $h_y = 0$ are dealt analogously.

4.1.3. Example 1: Implicit surface

Let $f(x, y, z) = 0$ be a smooth surface with continuous second derivatives of f. For a surface point $\mathbf{x}$ we have

(1) $h = 0$.
(2) $\nabla h = \nabla f / \|\nabla f\|$.
(3) The normal curvature of an implicit surface $f(\mathbf{x}) = 0$ is $\kappa_n = \mathbf{v} H_f \mathbf{v}^{\mathrm{T}} / \|\nabla f\|$ with unit tangent vector $\mathbf{v}$ and Hessian matrix H_f:

(a) If $f_x f_y f_z \neq 0$:

$$\kappa_1 = \frac{f_z^2 f_{yy} - 2 f_y f_z f_{yz} + f_y^2 f_{zz}}{\|\nabla f\|(f_y^2 + f_z^2)}, \quad \kappa_2 = \frac{f_z^2 f_{xx} - 2 f_x f_z f_{xz} + f_x^2 f_{zz}}{\|\nabla f\|(f_x^2 + f_z^2)},$$

$$\kappa_3 = \frac{f_y^2 f_{xx} - 2 f_x f_y f_{xy} + f_x^2 f_{yy}}{\|\nabla f\|(f_x^2 + f_y^2)}.$$

(b) If $f_z = 0$:

$$\kappa_1 = \frac{f_y^2 f_{xx} + f_x^2 f_{yy} + f_{zz} - 2 f_x f_y f_{xy} - 2 f_x f_{yz} + 2 f_y f_{xz}}{\|\nabla f\|(1 + f_x^2 + f_y^2)},$$

$$\kappa_2 = \frac{f_y^2 f_{xx} + f_x^2 f_{yy} + f_{zz} - 2 f_x f_y f_{xy} + 2 f_x^2 f_{yz} - 2 f_y f_{xz}}{\|\nabla f\|(1 + f_x^2 + f_y^2)},$$

$$\kappa_3 = \frac{f_{zz}}{\|\nabla f\|}.$$

H_h is determined by ∇h and the corresponding curvatures $\kappa_1, \kappa_2, \kappa_3$ (see formulas above).

4.1.4. Example 2: Parametrically defined surface

Let $\mathbf{x} = \mathbf{S}(u, v)$ be a smooth surface with continuous second derivatives. For a surface point $\mathbf{S} = (X, Y, Z)$ we have

(1) $h = 0$.

(2) $\nabla h = \dfrac{\mathbf{S}_u \times \mathbf{S}_v}{\|\mathbf{S}_u \times \mathbf{S}_v\|}$.

(3) (a) Case $h_x h_y h_z \neq 0$:

$$\kappa_1 = -\frac{L X_v^2 - 2 M X_u X_v + N X_u^2}{E X_v^2 - 2 F X_u X_v + G X_u^2}, \quad \kappa_2 = -\frac{L Y_v^2 - 2 M Y_u Y_v + N Y_u^2}{E Y_v^2 - 2 F Y_u Y_v + G Y_u^2},$$

$$\kappa_3 = -\frac{L Z_v^2 - 2 M Z_u Z_v + N Z_u^2}{E Z_v^2 - 2 F Z_u Z_v + G Z_u^2}.$$

(b) Case $h_z = 0$:

If $(X_u, X_v) \neq (0, 0)$ then $\mathbf{v}_1 = -X_v \mathbf{S}_u + X_u \mathbf{S}_v = (0, 0, \ldots)$ and

$$\kappa_1 = -\frac{L X_v^2 - 2 M X_u X_v + N X_u^2}{E X_v^2 - 2 F X_u X_v + G X_u^2}.$$

Let be $\mathbf{v}_0 := -Z_v \mathbf{S}_u + Z_u \mathbf{S}_v = (\ldots, \ldots, 0)$, and $\mathbf{v}_2 = (\mathbf{v}_0/\|\mathbf{v}_0\| + \mathbf{v}_1/\|\mathbf{v}_1\|)/\sqrt{2}$. Hence, $\|\mathbf{v}_2\| = 1$ and

$$\kappa_2 = -(L\xi_2^2 - 2M\xi_2\eta_2 + N\eta_2^2),$$

with

$$\xi_2 := -\frac{Z_v}{\sqrt{EZ_v^2 - 2FZ_uZ_v + GZ_u^2}} - \frac{X_v}{\sqrt{EX_v^2 - 2FX_uX_v + GX_u^2}},$$

$$\eta_2 := \frac{Z_u}{\sqrt{EZ_v^2 - 2FZ_uZ_v + GZ_u^2}} + \frac{X_u}{\sqrt{EX_v^2 - 2FX_uX_v + GX_u^2}}.$$

For $\mathbf{v}_3 = (-\mathbf{v}_0/\|\mathbf{v}_0\| + \mathbf{v}_1/\|\mathbf{v}_1\|)/\sqrt{2}$ we get $\kappa_3 = -(L\xi_3^2 - 2M\xi_3\eta_3 + N\eta_3^2)$, with

$$\xi_3 := \frac{Z_v}{\sqrt{EZ_v^2 - 2FZ_uZ_v + GZ_u^2}} - \frac{X_v}{\sqrt{EX_v^2 - 2FX_uX_v + GX_u^2}},$$

$$\eta_3 := -\frac{Z_u}{\sqrt{EZ_v^2 - 2FZ_uZ_v + GZ_u^2}} + \frac{X_u}{\sqrt{EX_v^2 - 2FX_uX_v + GX_u^2}}.$$

If $(X_u, X_v) = (0,0)$ then replace (X_u, X_v) by (Y_u, Y_v).

E, F, G are the coefficients of the first whereas L, M, N are the coefficients of the second fundamental form of the surface.

H_h is determined by ∇h and the curvatures $\kappa_1, \kappa_2, \kappa_3$ (see formulas above).

4.1.5. Example 3: Surfaces defined by several equations

In (Chuang and Hoffmann, 1990) algorithms for the computation of the surface normal and the normal curvature of surfaces defined by $m > 1$ equations are introduced. So the Hessian matrix of the normalform of such a surface can be evaluated by using the algorithm of (Chuang and Hoffmann, 1990) and the formulas derived above.

4.2. The first and second derivatives in the vicinity of the surface

Let $\mathbf{x} \in \Phi_0$ be in such a way that h is C^2-continuous with main curvatures $\kappa_{\min}, \kappa_{\max}$. The distance parameter $\delta \in \mathbb{R}$ is chosen in such a way that $1+\delta\kappa_{\min}(\mathbf{x}) > 0$, $1+\delta\kappa_{\max}(\mathbf{x}) > 0$ and h is C^2-continuous at point $\mathbf{x}_\delta := \mathbf{x} + \delta\nabla h(\mathbf{x})$. Differentiating the fundamental equation $h(\mathbf{x}_\delta) = h(\mathbf{x}) + \delta$ yields

$$\big(I + \delta H_h(\mathbf{x})\big)\nabla h(\mathbf{x}_\delta) = \nabla h(\mathbf{x}),$$

with Hessian matrix H_h and 3×3 unit matrix I. This is a linear system for vector $\nabla h(\mathbf{x}_\delta)$ with determinant $\det(I + \delta H_h(\mathbf{x})) = (1+\delta\kappa_{\min})(1+\delta\kappa_{\max}) \neq 0$. (Evaluate the characteristic polynomial $p(\lambda)$ of matrix δH_h for $\lambda = -1$.) The unique solution is

$$\boxed{\nabla h(\mathbf{x}_\delta) = \nabla h(\mathbf{x}).}$$

(For the proof take into account that $\nabla h(\mathbf{x})$ is eigenvector of H_h with eigenvalue 0.)

Differentiating this equation yields (as in the plane case)

$$\big(I + \delta H_h(\mathbf{x})\big)H_h(\mathbf{x}_\delta) = H_h(\mathbf{x}).$$

The unique solution of this linear system for $H_h(\mathbf{x}_\delta)$ is

$$\boxed{H_h(\mathbf{x}_\delta) = \frac{(1+\delta(\kappa_{\min}(\mathbf{x}) + \kappa_{\max}(\mathbf{x})))H_h(\mathbf{x}) - \delta H_h(\mathbf{x})^2}{(1+\delta\kappa_{\min}(\mathbf{x}))(1+\delta\kappa_{\max}(\mathbf{x}))}.}$$

(For the proof use the identity $-H_h^3 + (\kappa_{\min} + \kappa_{\max})H_h^2 - \kappa_{\min}\kappa_{\max}H_h = 0$.)

If $\kappa_{\min} = \kappa_{\max} = \kappa$ (umbilic case) we get the simplification

$$H_h(\mathbf{x}_\delta) = \frac{H_h(\mathbf{x})}{1 + \delta\kappa(\mathbf{x})}.$$

(In this case the minimal polynomial of H_h is $\lambda^2 - \kappa\lambda = 0$.)

A second possibility for determining $H_h(\mathbf{x}_\delta)$ is using the relations

$$\kappa_1(\mathbf{x}_\delta) = \frac{\kappa_1(\mathbf{x})}{1 + \delta\kappa_1(\mathbf{x})}, \quad \kappa_2(\mathbf{x}_\delta) = \frac{\kappa_2(\mathbf{x})}{1 + \delta\kappa_2(\mathbf{x})}, \quad \kappa_3(\mathbf{x}_\delta) = \frac{\kappa_3(\mathbf{x})}{1 + \delta\kappa_3(\mathbf{x})}$$

and the formulas for $h_{xx}, h_{xy}, \ldots$ above.

Remark. For the offset surface $h = \delta$ we get the following results:

- $$\kappa_{\min}(\mathbf{x}_\delta) = \frac{\kappa_{\min}(\mathbf{x})}{1 + \delta\kappa_{\min}(\mathbf{x})}, \quad \kappa_{\max}(\mathbf{x}_\delta) = \frac{\kappa_{\max}(\mathbf{x})}{1 + \delta\kappa_{\max}(\mathbf{x})}.$$

  (Corresponding principal directions are parallel!)
- Gauss curvature

  $$K(\mathbf{x}_\delta) = \frac{K(\mathbf{x})}{1 + 2\delta H(\mathbf{x}) + \delta^2 K(\mathbf{x})},$$

  mean curvature

  $$H(\mathbf{x}_\delta) = \frac{H(\mathbf{x}) + \delta K(\mathbf{x})}{1 + 2\delta H(\mathbf{x}) + \delta^2 K(\mathbf{x})}.$$

4.3. On the linkage curve theorem

In order to show that the normalform is an appropriate tool for theoretical considerations we give a simple proof of the linkage curve theorem of X. Ye.

At a surface point $\mathbf{p}$ we will use a *local coordinate system* in such a way that $\mathbf{p}$ is the origin $\mathbf{0}$ and $\nabla h(\mathbf{0}) = (0, 0, 1)$. Hence the x-y-plane is the tangent plane at point $\mathbf{p} = \mathbf{0}$ and for the Hessian we get the simplification (because of Eqs. (1)–(3) in Section 4.1)

$$H_h(\mathbf{0}) = \begin{pmatrix} h_{xx} & h_{xy} & 0 \\ h_{yx} & h_{yy} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

For this specialization we get the following results

- H_h is determined uniquely by the normal curvatures of any *three* tangential directions. This is the 3-Tangent-Theorem (cf. Pegna and Wolter, 1992). The proof is straight forwarded calculation.
- h_{xx}, h_{yy} are the normal curvatures in direction $(1, 0, 0), (0, 1, 0)$ respectively, $H = (h_{xx} + h_{yy})/2$ the mean curvature and $K = h_{xx}h_{yy} - h_{xy}^2$ the Gaussian curvature.

- From the Taylor expansion of h at point $\mathbf{0}$ we get the *Dupin indicatrix* of the surface at point $\mathbf{0}$:

$$h_{xx}x^2 + 2h_{xy}xy + h_{yy}y^2 = \pm 1.$$

Lemma. *Let Φ_1, Φ_2 be two G^2-continuous surfaces with common point $\mathbf{0}$, common tangent plane $z = 0$ and common tangent planes along the common smooth curve Γ: $\mathbf{x}(t) = (t, b(t), c(t))$, $t \in [0, t_0]$, with $\mathbf{x}(0) = \mathbf{0}$ and $\dot{\mathbf{x}}(0) = (1, 0, \dot{c})$. We then get for the Hessian matrices H_{h_1}, H_{h_2} of the normalform functions h_1, h_2 of the surfaces Φ_1, Φ_2: $h_{1,xx}(\mathbf{0}) = h_{2,xx}(\mathbf{0})$, $h_{1,xy}(\mathbf{0}) = h_{2,xy}(\mathbf{0})$.*

Proof. For $(t, b(t), c(t)) \in \Gamma$ we have $\nabla h_1(t, b(t), c(t)) = \nabla h_2(t, b(t), c(t))$. Differentiating this equation yields

$$H_{h_1}\big(t, b(t), c(t)\big)\big(1, \dot{b}(t), \dot{c}(t)\big)^{\mathrm{T}} = H_{h_2}\big(t, b(t), c(t)\big)\big(1, \dot{b}(t), \dot{c}(t)\big)^{\mathrm{T}}.$$

For $t = 0$ we get $h_{1,xx}(\mathbf{0}) = h_{2,xx}(\mathbf{0})$, $h_{1,xy}(\mathbf{0}) = h_{2,xy}(\mathbf{0})$ (Remember the simple Form of the Hessian matrix and $\dot{b}(0) = 0$ at point $\mathbf{0}$!). □

We obviously get for the Hessian matrices of the surfaces Φ_1, Φ_2 in the lemma shown above the stronger result $H_{h_1}(\mathbf{0}) = H_{h_2}(\mathbf{0})$ if we choose additionally a suitable geometric restriction which yields $h_{1,yy}(\mathbf{0}) = h_{2,yy}(\mathbf{0})$. For example *one* of the following conditions

- The equality of the mean curvatures:

$$2H(\mathbf{0}) = h_{1,xx}(\mathbf{0}) + h_{1,yy}(\mathbf{0}) = h_{2,xx}(\mathbf{0}) + h_{2,yy}(\mathbf{0}).$$

- The equality of the Gaussian curvatures:

$$K(\mathbf{0}) = h_{1,xx}(\mathbf{0})h_{1,yy}(\mathbf{0}) - h_{1,xy}(\mathbf{0})^2 = h_{2,xx}(\mathbf{0})h_{2,yy}(\mathbf{0}) - h_{2,xy}(\mathbf{0})^2$$

  if the normal curvature $h_{1,xx}(\mathbf{0})$ in direction $(1, 0, 0)$ does not vanish.
- The equality of the normal curvature in a direction $\mathbf{v} = (a, 1, 0)$, $a \in \mathbb{R}$, transversal to curve Γ at point $\mathbf{0}$:

$$\kappa_a = \frac{a^2 h_{1,xx} + h_{1,yy} + 2a h_{1,xy}}{1 + a^2} = \frac{a^2 h_{2,xx} + h_{2,yy} + 2a h_{2,xy}}{1 + a^2}.$$

These considerations prove the following theorem

Linkage Curve Theorem (Ye, 1996). *Let Φ_1 and Φ_2 be two G^2-continuous surfaces which are tangent plane continuous along a smooth linkage curve Γ. Φ_1 and Φ_2 are G^2-continuous along Γ if* one *of the following conditions is fulfilled*:

- *Φ_1 and Φ_2 have the same* mean curvature *along Γ.*
- *Φ_1 and Φ_2 have the same* Gaussian curvature *and nonvanishing normal curvature in Γ-direction along Γ.*
- *There exists a transversal* vector field (*not necessarily continuous*) *along Γ, such that the normal curvatures of Φ_1 and Φ_2 in directions of the vector field are the same.*

5. Applications

5.1. Stable first order foot point algorithms

For applying the normalform and its first two derivatives it is essential to have stable algorithms for determining foot points on curves and surfaces. We give here algorithms for implicit and parametric curves which use only first order derivatives. They can easily be extended to surfaces. The heart of the algorithms is the combination of calculating foot points on tangents and approximate foot points on tangent parabolas. The curvature of the curves (surfaces) is respected indirectly by the tangent parabolas.

5.1.1. Foot point algorithms for parametric curves and surfaces

Let Γ: $\mathbf{x}(t) = \mathbf{c}(t)$ be a smooth planar curve, $\mathbf{p}$ a point in the vicinity of Γ and t_0 the parameter of a starting point for the *foot point algorithm*:

repeat
$\mathbf{p}_i = \mathbf{c}(t_i)$,
$\Delta t = (\mathbf{p} - \mathbf{p}_i) \cdot \dot{\mathbf{c}}(t_i)/\dot{\mathbf{c}}(t_i)^2$, $\quad \mathbf{q}_i = \mathbf{p}_i + \Delta t \dot{\mathbf{c}}(t_i)$ (foot point on tangent),
$\mathbf{p}_{i+1} = \mathbf{c}(t_i + \Delta t)$, $\mathbf{f}_1 := \mathbf{q}_i - \mathbf{p}_i$, $\mathbf{f}_2 := \mathbf{p}_{i+1} - \mathbf{q}_i$,
if $\|\mathbf{q}_i - \mathbf{p}_i\| > \varepsilon$ **then** (one Newton step for the foot point on the tangent parabola $\mathbf{x} = \mathbf{p}_i + \alpha\mathbf{f}_1 + \alpha^2\mathbf{f}_2$)
$a_0 := (\mathbf{p} - \mathbf{p}_i) \cdot \mathbf{f}_1$, $a_1 := 2\mathbf{f}_2 \cdot (\mathbf{p} - \mathbf{p}_i) - \mathbf{f}_1^2$, $a_2 := -3\mathbf{f}_1 \cdot \mathbf{f}_2$, $a_3 := -2\mathbf{f}_2^2$,
$$\alpha := 1 - \frac{a_0 + a_1 + a_2 + a_3}{a_1 + 2a_2 + 3a_3},$$
if $0 < \alpha < \alpha_{\max}$ **then** (prevent extreme cases)
$t_{i+1} = t_i + \alpha \Delta t$, $\mathbf{p}_{i+1} = \mathbf{c}(t_{i+1})$
until $\|\mathbf{p}_i - \mathbf{p}_{i+1}\| < \varepsilon$.
foot point $\mathbf{f} = \mathbf{p}_{i+1}$.

For the examples below we set $\varepsilon = 10^{-6}$ and $\alpha_{\max} = 20$.

We get the analogous algorithm for a parametric *surface* Φ: $\mathbf{x} = \mathbf{S}(u, v)$, in case we replace Δt by the corresponding parameter corrections $(\Delta u, \Delta v)$ for the foot point on the tangent plane at point $\mathbf{p}_i = \mathbf{S}(u_i, v_i)$. They are the solution of the linear system:

$$(\mathbf{p} - \mathbf{p}_i) \cdot \mathbf{e}_u = \Delta u \mathbf{e}_u^2 + \Delta v (\mathbf{e}_u \cdot \mathbf{e}_v),$$
$$(\mathbf{p} - \mathbf{p}_i) \cdot \mathbf{e}_v = \Delta u (\mathbf{e}_u \cdot \mathbf{e}_v) + \Delta v \mathbf{e}_v^2,$$

with $\mathbf{e}_u := \mathbf{S}_u(u_i, v_i)$, $\mathbf{e}_v := \mathbf{S}_v(u_i, v_i)$.

5.1.2. Foot point algorithms for implicit curves and surfaces

Let Γ: $f(\mathbf{x}) = 0$ be a smooth planar implicit curve. We use the following procedure `curvepoint` which calculates for a given point $\mathbf{p}$ in the vicinity of Γ a curve point $\mathbf{c}$ along the steepest way:

(CP0) $\mathbf{q}_0 = \mathbf{p}$
(CP1) **repeat** $\mathbf{q}_{k+1} = \mathbf{q}_k - (f(\mathbf{q}_k)/\nabla f(\mathbf{q}_k)^2)\nabla f(\mathbf{q}_k)$ (Newton step)
until $\|\mathbf{q}_{k+1} - \mathbf{q}_k\| < \varepsilon$.
curve point $\mathbf{c} = \mathbf{q}_{k+1}$.

Let $\mathbf{p}$ be a point in the vicinity of curve Γ. The following algorithm determines the *foot point* of $\mathbf{p}$ on Γ:

(FP0) $\mathbf{p}_0 = \text{curvepoint}(\mathbf{p})$

(FP1) **repeat**

$\mathbf{q}_i = \mathbf{p} - ((\mathbf{p}-\mathbf{p}_i)\cdot\nabla f(\mathbf{p}_i)/\nabla f(\mathbf{p}_i)^2)\nabla f(\mathbf{p}_i)$ (foot point on tangent line),

$\mathbf{p}_{i+1} = \texttt{curvepoint}(\mathbf{q}_i)$, $\mathbf{f}_1 := \mathbf{q}_i - \mathbf{p}_i$, $\mathbf{f}_2 := \mathbf{p}_{i+1} - \mathbf{q}_i$,

if $\|\mathbf{q}_i - \mathbf{p}_i\| > \varepsilon$ **then** (one Newton step for the foot point on the tangent parabola $\mathbf{x} = \mathbf{p}_i + \alpha\mathbf{f}_1 + \alpha^2\mathbf{f}_2$)

$a_0 := (\mathbf{p}-\mathbf{p}_i)\cdot\mathbf{f}_1$, $a_1 := 2\mathbf{f}_2\cdot(\mathbf{p}-\mathbf{p}_i) - \mathbf{f}_1^2$, $a_2 := -3\mathbf{f}_1\cdot\mathbf{f}_2$, $a_3 := -2\mathbf{f}_2^2$,

$$\alpha := 1 - \frac{a_0 + a_1 + a_2 + a_3}{a_1 + 2a_2 + 3a_3},$$

if $0 < \alpha < \alpha_{\max}$ **then** (prevent extreme cases)

$\mathbf{q}_i = \mathbf{p}_i + \alpha\mathbf{f}_1 + \alpha^2\mathbf{f}_2, \quad \mathbf{p}_{i+1} = \text{curvepoint}(\mathbf{q}_i)$,

until $\|\mathbf{p}_i - \mathbf{p}_{i+1}\| < \varepsilon$.

foot point $\mathbf{f} = \mathbf{p}_{i+1}$.

For the analogous *surface* foot point algorithm one has to replace only the words “curve” and “line” by “surface” and “plane”.

For *displaying* an implicit curve we use the following tracing algorithm for a smooth implicit curve $\Gamma\colon\ f = 0$:

(IC0) Choose a starting point $\mathbf{q}_1$ in the vicinity of Γ.

(IC1) $\mathbf{p}_i := \text{curvepoint}(\mathbf{q}_i)$ (see algorithm curvepoint above),

$\mathbf{t}_i := (-f_y(\mathbf{p}_i), f_x(\mathbf{p}_i))/\|\ldots\|$ (unit tangent),

$\mathbf{q}_{i+1} := \mathbf{p}_i + \delta\mathbf{t}_i$ (δ: step length).

The tracing algorithm stops if $\mathbf{p}_i$ is “near” a prescribed endpoint (or another termination).

5.2. *Curvature of bisectors of two curves*

Let Γ_1, Γ_2 be two smooth parametric or implicit curves and $h_1 = 0$, $h_2 = 0$ their normal forms. Hence, the equations

$$h_1(\mathbf{x}) - h_2(\mathbf{x}) = 0, \qquad h_1(\mathbf{x}) + h_2(\mathbf{x}) = 0$$

are implicit representations of the *bisector curves* (points, which have equal distance to Γ_1 and Γ_2, cf. (Farouki and Johnstone, 1994; Hartmann, 1998b)).

The bisectors can be traced by the marching algorithm for implicit curves above.

Example. Fig. 1 shows two Bézier curves (bold curves) and their bisectors. The curvature κ at a curve point $\mathbf{x}$ is visualized by a pin proportional to κ (*porcupines*). The necessary evaluation of the normalforms of the Bézier curves and their derivatives is done by the foot point algorithm (Section 5.1.1) and the formulas for the gradient and the Hessian matrix of Section 3.

5.3. *Curvature of planar G^n-blending curves*

Let be $\Gamma_1\colon\ f_1(\mathbf{x}) = 0$, $\Gamma_2\colon\ f_2(\mathbf{x}) = 0$ two implicit curves and $\Gamma_0\colon\ f_0(\mathbf{x}) = 0$ a line which intersects Γ_1 and Γ_2 (f_i differentiable enough). Then

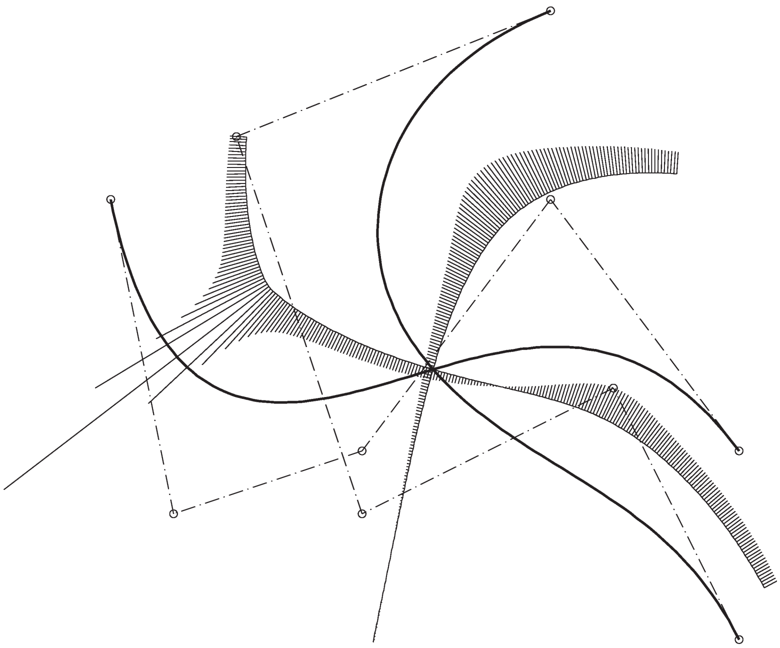

Fig. 1. Curvature of the bisectors of two Bézier curves.

$$\Phi_\mu\colon\ (1-\mu) f_1 f_2 - \mu f_0^{n+1} = 0, \quad 0 < \mu < 1,$$

is for any μ a curve with G^n-continuous contact to the curves Γ_1 and Γ_2 at the intersection points $\Gamma_1 \cap \Gamma_0$, $\Gamma_2 \cap \Gamma_0$. The blending curves Φ_μ are called *parabolic functional splines* (Hartmann, 1998b)).

Example. By using the normal form this (and any other) implicit blending method is applicable to nearly arbitrary curves, especially to parametric curves. Fig. 2 shows a G^2-blending curve of two Bézier curves. The continuity of the curvature between the Bézier curves and the blending curve is visualized by pins representing the magnitude of the curvature.

5.4. Isophotes on G^n-blending surfaces

Let $\Phi_1\colon\ f_1 = 0$, $\Phi_2\colon\ f_2 = 0$ be two implicit surfaces and Γ the plane implicit curve (the *correlation curve*)

$$k(c,d) = (1-\mu)\frac{cd}{c_0 d_0} - \mu\left(1 - \frac{c}{c_0} - \frac{d}{d_0}\right)^{n+1} = 0, \quad 0 < \mu < 1,\ n \geqslant 0.$$

Then the implicit surface $\Sigma\colon\ F(\mathbf{x}) := k(f_1(\mathbf{x}), f_2(\mathbf{x})) = 0$ has G^n-continuous contact to the surfaces Φ_1 and Φ_2 (cf. Hartmann, 1998a). We call such blending surfaces *elliptic*

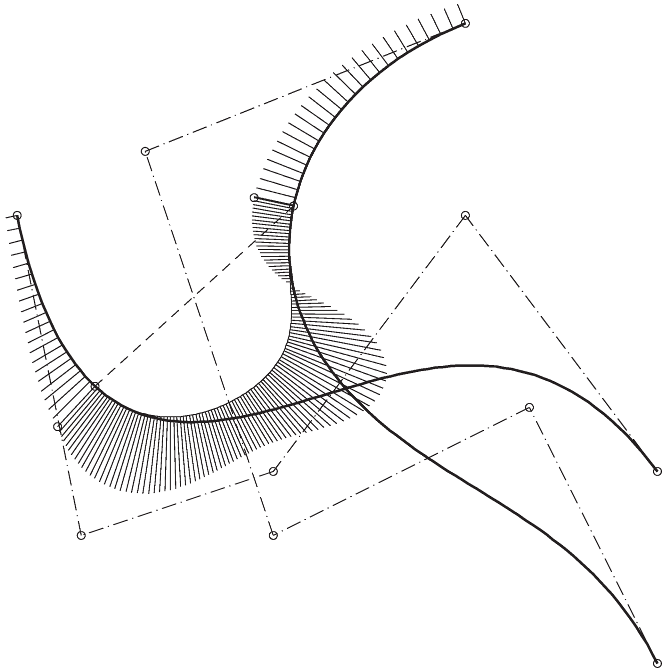

Fig. 2. Curvature of a G^2-blending of two Bézier curves.

functional splines. (For $n = 1$, $\mu = 1/3$ the correlation curve $k = 0$ is an ellipse that touches the coordinate axes.)

Example. Let Φ_1 and Φ_2 be two tensor product Bézier patches with parametric representations

$$\Phi_1\colon\ \mathbf{x} = \bigl(10v - 5, 10u - 5, 6(u - u^2 + v - v^2)\bigr),$$
$$\Phi_2\colon\ \mathbf{x} = \bigl(6(u - u^2 + v - v^2), 10u - 5, 10v - 5\bigr).$$

and normal forms $h_1 = 0$ and $h_2 = 0$ respectively. The normalforms are not explicitly known. For visualizing the blending surface $F = k(h_1, h_2)$ and its curvature we use the formulas for ∇h_i and H_{h_i} developed above. Fig. 3 shows a G^2-blending (normal curvatures are continuous), Fig. 4 a G^1-blending (only tangent planes are continuous) with parameters $c_0 = d_0 = 2.5$ and $\mu = 0.1$. The curvature is visualized by *isophotes*. An isophote is the collection of all surface points for which the scalarproduct $\mathbf{n} \cdot \mathbf{v}$ is constant for unit normal $\mathbf{n}$ and fixed unit vector $\mathbf{v}$ (light vector). For an implicit surface $f = 0$ the isophotes are intersection curves between the given surface and the implicit surface with the equation $\nabla f / \|\nabla f\| \cdot \mathbf{v} - c = 0$, $-1 < c < 1$. They can be traced by the algorithm given in (Bajaj et al., 1988) or (Hartmann, 1998a). Fig. 4 shows the tangent discontinuity of the isophotes (and the normal curvatures) at the curves of contact between the blending surface and the Bézier patches. Light vector $\mathbf{v}$ for the figures is $(2, 2, 5)/\| \cdots \|$.

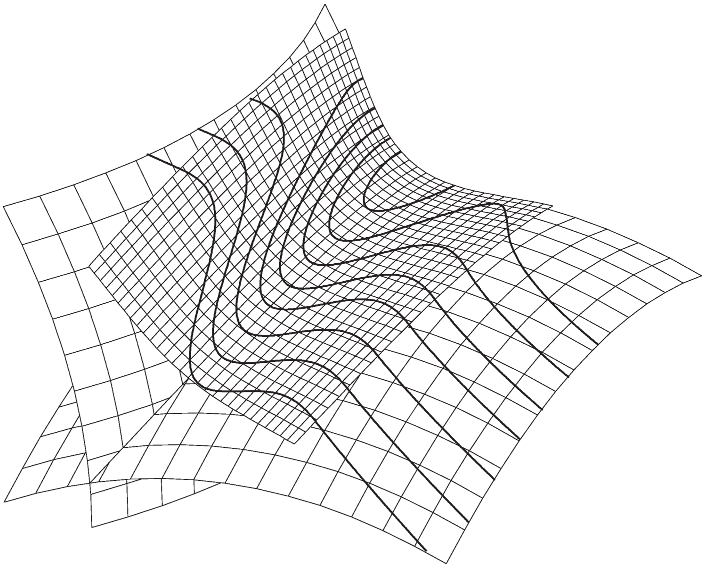

Fig. 3. Isophotes of a G^2-blending of two tensor product Bézier surfaces.

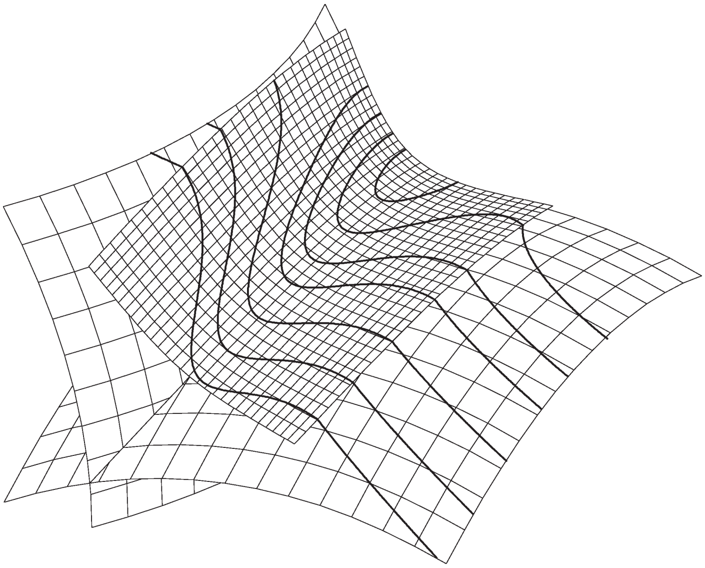

Fig. 4. Isophotes of a G^1-blending of two tensor product Bézier surfaces.

5.5. Isophotes, curvature lines and feature lines of a smooth approximation of a set of intersecting surfaces

Given are

(1) the implicit surface Φ_1: $(x-2)^4 + y^4 - r_1^4 = 0$, $r_1 = 2$,

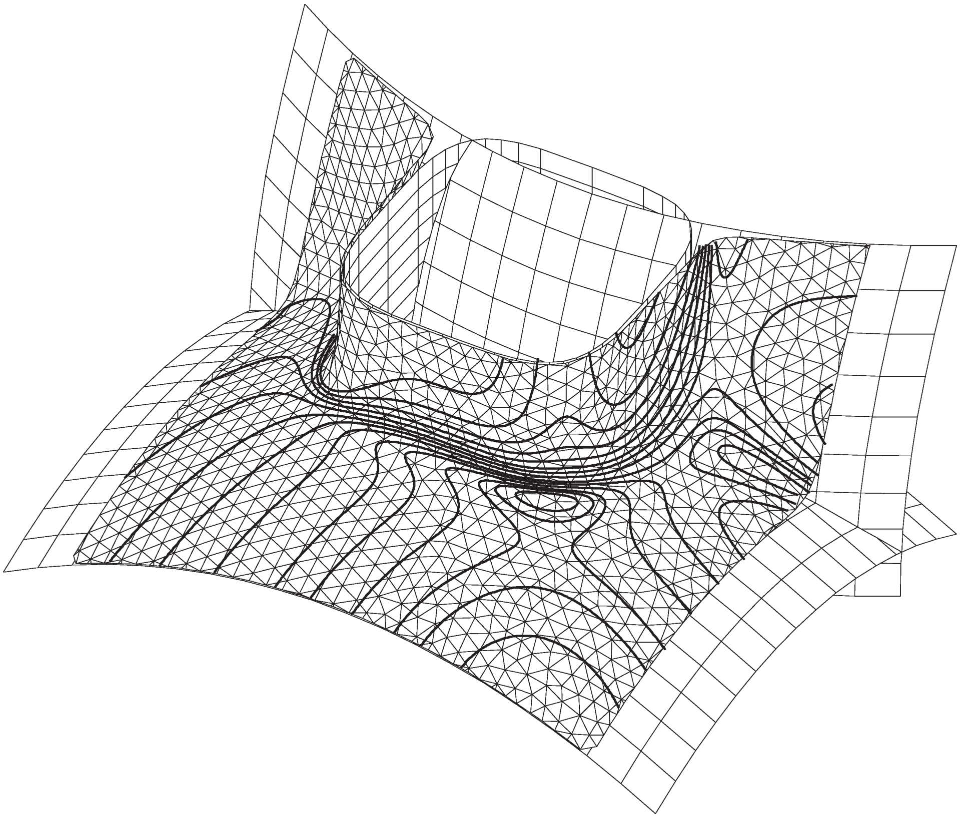

Fig. 5. Isophotes on a smooth approximation of three surfaces.

(2) the parametric surface patch

$$\Phi_2\colon\ \mathbf{x} = \big(10v - 5, 10u - 5, 6(u - u^2 + v - v^2)\big), \quad 0 \leqslant u \leqslant 1,\ 0 \leqslant v \leqslant 0.8,$$

(3) the parametric surface patch

$$\Phi_3\colon\ \mathbf{x} = \big(6(u - u^2 + v - v^2) - 5, 10u - 5, 10v - 5\big),$$
$$0 \leqslant u \leqslant 1,\ 0.5 \leqslant v \leqslant 1.$$

Let $h_1(\mathbf{x}) = 0$, $h_2(\mathbf{x}) = 0$, $h_3(\mathbf{x}) = 0$ be the normalforms of these surfaces. The implicit surface $\Phi\colon\ f(\mathbf{x}) := h_1(\mathbf{x})h_2(\mathbf{x})h_3(\mathbf{x}) = c$, $c > 0$ is a smooth approximation of the set of surfaces Φ_1, Φ_2, Φ_3. The approximation is independent of their representations.

Fig. 5 shows a triangulation of $f(\mathbf{x}) = c$ for $c = 0.2$ and some isophotes for light vector $\mathbf{v} = (2, 2, 5)/\| \cdots \|$. The triangulation is obtained by applying the marching method introduced in (Hartmann, 1998c) for implicit surfaces.

A *curvature line* Γ is a surface curve with the following property: The tangent of Γ at a non umbilical surface point $\mathbf{x}$ is one of the principal directions of the normal curvatures. Fig. 6 shows the given surface together with some lines of curvature.

For displaying the *curvature line* we use the following tracing algorithm:

(CL0) $\mathbf{q}_0$ point in the vicinity of the surface, δ step length.
$\mathbf{p}_0 = \text{surfacepoint}(\mathbf{q}_0)$ (see Section 5.1.2).
Calculate the principal directions $\mathbf{v}_{\min}, \mathbf{v}_{\max}$ at point $\mathbf{p}_0$ (see Section 4.1).
Let $\mathbf{v}_0$ be one of the four directions $\mathbf{v}_{\min}, -\mathbf{v}_{\min}, \mathbf{v}_{\max}, -\mathbf{v}_{\max}$.

(CL1) Determine the intermediate point $\mathbf{p}'_{i+1} := \text{surfacepoint}(\mathbf{p}_i + \delta\mathbf{v}_i)$ and its principal directions $\mathbf{v}'_{\min}, \mathbf{v}'_{\max}$.
Let be $c_1 := \mathbf{v}'_{\min} \cdot \mathbf{v}_i$, $c_2 := \mathbf{v}'_{\max} \cdot \mathbf{v}_i$.
If $|c_1| > 0.7$ then $\mathbf{v}'_{i+1} := \text{sign}(c_1)\mathbf{v}'_{\min}$ else $\mathbf{v}'_{i+1} := \text{sign}(c_2)\mathbf{v}'_{\max}$.
$\mathbf{t}_i := \frac{1}{2}(\mathbf{v}_i + \mathbf{v}'_{i+1})$ (tracing direction).
Determine $\mathbf{p}_{i+1} = \text{surfacepoint}(\mathbf{p}_i + \delta\mathbf{t}_i)$ and its principal directions $\mathbf{v}_{\min}, \mathbf{v}_{\max}$.
Let be $c_1 := \mathbf{v}_{\min} \cdot \mathbf{t}_i$, $c_2 := \mathbf{v}_{\max} \cdot \mathbf{t}_i$.
If $|c_1| > 0.7$ then $\mathbf{v}_{i+1} := \text{sign}(c_1)\mathbf{v}_{\min}$ else $\mathbf{v}_{i+1} := \text{sign}(c_2)\mathbf{v}_{\max}$.

Remark. For many purposes the global error of the curvature line algorithm may be small enough. But it can be further reduced by RUNGE–KUTTA tracing steps.

For various applications (segmentation, pattern recognition, ... , cf. (Belyaev et al., 1997; Belyaev et al., 1998; Lukacs and Andor, 1998)) *feature lines* are important. Feature lines are either *ridges* or *ravines*. A ridge on a regular surface consists of the local positive maxima of the maximal main curvature along its associated curvature line. A ravine consists of the local positive minima of the minimal main curvature along its associated curvature line. They can be traced from a starting point on a starting curvature line to the local extremum on an associated curvature line in the neighborhood using divided differences instead of derivatives of the curvature. Fig. 6 contains some feature lines (thick curves) of the approximation surface.

5.6. *Curvature of an intersection curve*

Let be Φ_1: $f_1(\mathbf{x}) = 0$, Φ_2: $f_1(\mathbf{x}) = 0$ two intersecting implicit surfaces which are differentiable enough. The curvature at a point of the intersection curve Γ is the length of the following vector (cf. (Hartmann, 1996))

$$\mathbf{c}_\kappa := \frac{\alpha_1 \nabla f_2 - \alpha_2 \nabla f_1}{\|\nabla f_1 \times \nabla f_2\|^3}$$

with

$$\alpha_1 := (\nabla f_1 \times \nabla f_2) H_{f_1} (\nabla f_1 \times \nabla f_2)^{\mathrm{T}}, \quad \alpha_2 := (\nabla f_1 \times \nabla f_2) H_{f_2} (\nabla f_1 \times \nabla f_2)^{\mathrm{T}}.$$

Fig. 7 shows the intersection curve of the blending surface of two Bézier patches (see Section 5.4) with a cylinder. By using the implicit representation of the blending surface given in Section 5.4, the implicit representation of the cylinder and the formula for the curvature above the curvature at a curve point $\mathbf{p}$ is calculated and visualized by a circle orthogonal to the intersection curve with midpoint $\mathbf{p}$ and radius proportional to the curvature $\kappa(\mathbf{p})$.

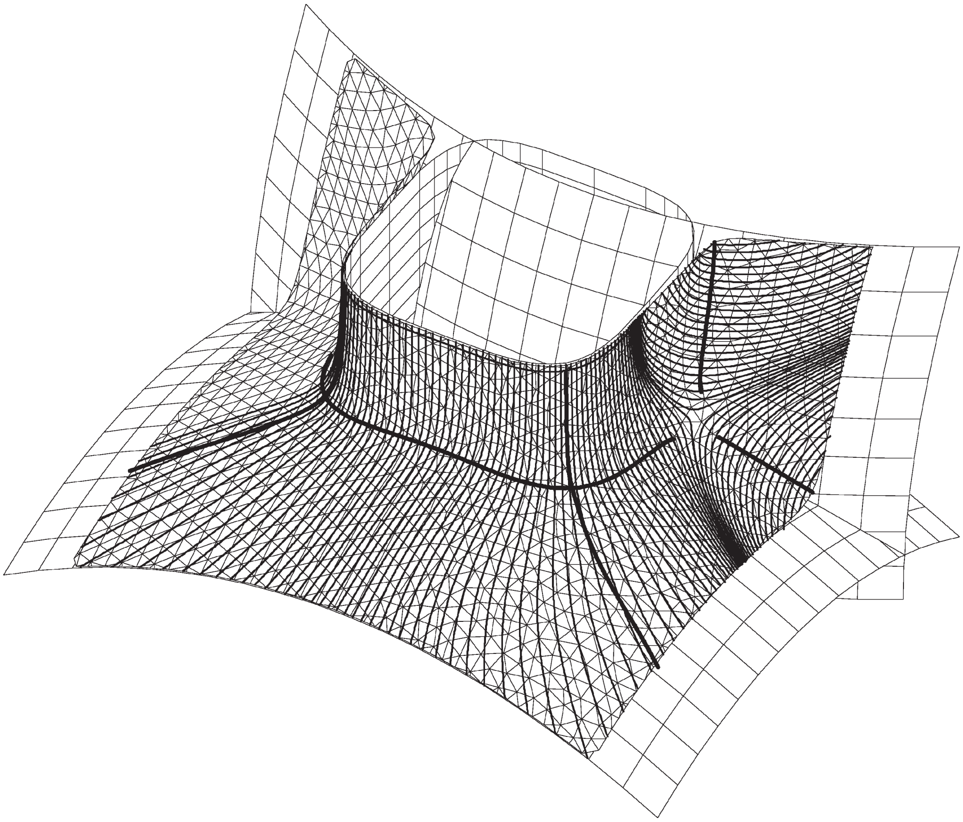

Fig. 6. Curvature lines and feature lines (thick) on a smooth approximation of three surfaces.

Remark. Out of the formula for the normal curvature of an implicit surface (see Section 4.1) we get the following formula for the curvature at a point of the intersection curve

$$\kappa = \frac{\sqrt{\kappa_1^2 + \kappa_2^2 - 2\kappa_1\kappa_2 \cos\varphi}}{\sin\varphi}$$

with

(1) the normal curvature κ_i on surface Φ_i for direction of the intersection curve,

(2) the (smaller) angle φ between the tangent planes of the surfaces.

This formula is independent on the representations of the surfaces!

$\kappa_1' := \kappa_1/\sin\varphi$ is the curvature of the intersection curve between surface Φ_1 and the tangent plane of surface Φ_2, $\kappa_2' := \kappa_2/\sin\varphi \ldots$.

For the *geodesic curvatures* κ_{1g}, κ_{2g} (of the intersection curve) on surface Φ_1, Φ_2 respectively one gets the formulas (cf. (Hartmann, 1996)):

$$\kappa_{1g} = \frac{\kappa_1 \cos\varphi - \kappa_2}{\sin\varphi}, \quad \kappa_{2g} = \frac{\kappa_1 - \kappa_2 \cos\varphi}{\sin\varphi}.$$

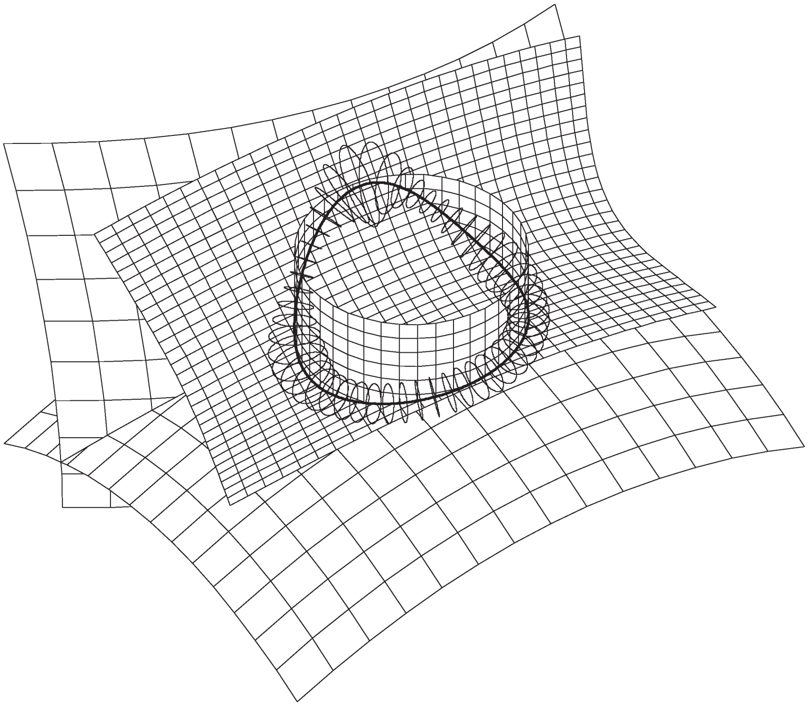

Fig. 7. Curvature of an intersection curve of a blending surface and a cylinder.

6. Conclusion

Formulas for the second derivatives of the normalform function are derived and applied to the visualization of the curvature of curves and surfaces defined by normal forms. The evaluation of the normalform function and its derivatives needs the numerical determination of foot points. Suitable algorithms for curves and surfaces are introduced and applied to examples. The normal form is an appropriate tool for proving theoretical results.

References

Bajaj, C.L., Hoffmann, C.M., Lynch, R.E. and Hopcroft, J.E.H. (1988), Tracing surface intersections, Computer Aided Geometric Design 5, 285–307.

Belyaev, A.G., Bogaevski, I.A. and Kunii, T.L. (1997), Ridges and ravines on a surface and segmentation of range images, in: *Vision Geometry VI*, Proc. SPIE 3168, Melter, R.A., Wu, A.Y., Latecki, L.J., eds., 106–114.

Belyaev, A.G., Pasko, A.A. and Kunii, T.L. (1998), Ridges and ravines on implicit surfaces, in: *Proc. Computer Graphics International* 1998, Wolter, F.-E., Patrikalakis, N.M., eds., 530–535.

Chuang, J.-H. and Hoffmann, C.M. (1990), Curvature computations on surfaces in n-space, Mathematical Modelling and Numerical Analysis 26, 95–112.

Courant, R. and Hilbert, D. (1962), *Methods of Mathematical Physics II*, Interscience Publishers, J. Wiley, New York.

Farouki, R.T. and Johnstone, J.K. (1994), Computing point/curve and curve/curve bisectors, in: *Design and Application of Curves and Surfaces: Mathematics of Surfaces V*, Fisher, R.B., ed., Oxford University Press, 327–354.

Farouki, R.T. (1998), On integrating lines of curvature, Computer Aided Geometric Design 15, 187–192.

Gilbarg, D. and Trudinger, N.S. (1983), *Elliptic Partial Differential Equations of Second Order*, Springer, Berlin.

Hartmann, E. (1996), G^2-interpolation and blending on surfaces, The Visual Computer 12, 181–192.

Hartmann, E. (1998a), Numerical implicitization for intersection and G^n-continuous blending of surfaces, Computer Aided Geometric Design 15, 377–397.

Hartmann, E. (1998b), The normalform of a planar curve and its application to curve design, in: *Mathematical Methods for Curves and Surfaces II*, Daehlen, M., Lyche, T., Schumaker, L., eds., Vanderbilt University Press, Nashville, 237–244.

Hartmann, E. (1998c), A marching method for the triangulation of surfaces, The Visual Computer 14, 95–108.

Hermann, T. (1998), On the smoothness of offset surfaces, Computer Aided Geometric Design 15, 529–533.

Lukacs, G. and Andor, L. (1998), Computing natural division lines on free-form surfaces, in: *Mathematical Methods for Curves and Surfaces II*, Daehlen, M., Lyche, T. and Schumaker, L., eds., Vanderbilt University Press, Nashville.

Ostrowski, A. (1956), Zur Theorie der partiellen Differentialgleichungen erster Ordnung, Math. Zeitschr. 66, 70–87.

Pegna, J. and Wolter, F.E. (1992), Geometrical criteria to guarantee curvature continuity of blend surfaces, Transactions of ASME, Journal of Mechanical Design 114, 201–210.

Pottmann, H. and Opitz, K. (1994), Curvature analysis and visualization for functions defined on Euclidean spaces or surfaces, Computer Aided Geometric Design 11, 655–674.

Weise, K.H. (1966), *Differentialgleichungen*, Vandenhoek & Ruprecht, Göttingen.

Ye, X. (1996), The Gaussian and mean curvature criteria for curvature continuity between surfaces, Computer Aided Geometric Design 13, 549–567.