An introduction to the eXtended Finite Element Method (X-FEM)
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Extended Finite Elements

Lecture plan

- Introduction
- Reminder
- Simple problems (jump on the primal variable)
- Extensions in 2D / 3D
- Other types of problems (jump on the derivatives)
- Other applications and current research
- Boundary conditions
- References
Extended Finite Elements

Course Notes available at:

http://www.cgeo.ulg.ac.be/X-FEM
“Classical” finite element computation

- The geometry is bounded by element sides
  - Bounds the computation domain
  - Bounds the interface between zones of dissimilar properties
  - A change in geometry implies a change in the mesh
  - Time evolving problems may induce remeshing at each time step in the computation
Extended Finite Elements

Introduction

- Mesh generation techniques
  - May be costlier than the sole finite element computation
  - (Often) necessitates a strong human interaction
  - Are a potential source of mistakes
    - Of human origin
    - Or from the lack of robustness of remeshing algorithms
Extended Finite Elements

Introduction
Extended Finite Elements

Introduction

- The idea here:
  - Minimize the constraints on the mesh that is used in the FEM simulations
  - However, mesh generations is still necessary
    - e.g. the accuracy of the computation depends on the quality of the mesh
      → mesh adaptation
We will rely on the classical FEM; starting with the weak form of a physical problem:

Find $u \in H^1(\Omega)$ such that

$$\int_{\Omega} a(u, v) \, d\Omega = \int_{\Omega} b(v) \, d\Omega \quad \forall \, v \in H^1_0(\Omega)$$

Discretization: One looks for $u$ in a discrete function space $V_h \subset H^1(\Omega)$ (trial functions $v$ belong to the same space $V_{0h} \subset H^1_0(\Omega)$)

$$u_h(x) = \sum_i \lambda_i N_i(x) \quad , \quad x \in \Omega$$
A space-conforming mesh is used to define the shape functions SFs $u(x) = \sum_k \lambda_k N_k$ for $x \in T_j$

- They have a compact support
- Partition of unity $\sum N_i = 1$
- Interpolation $u(x_i) = \lambda_i$
Extended Finite Elements

Reminder

- SFs with a compact support
  - Allows to have banded matrices (low memory imprint)
- Partition of unity
  - One is able to represent a constant field!
- Interpolation
  - Easy to impose Dirichlet boundary conditions
- Use of conforming meshes
  - Pre-computations of many operators is possible at an elementary level
Extended Finite Elements

Simple problem

- Clamped 1D Bar (L, E, S) with a variable load \( f(x) \)
- One wants to get the displacement \( u(x) \) and assume that the bar is cut at some place
  - With the classical FEM
  - With the eXtended Finite Element Method

\[ f(x) \]
Extended Finite Elements

Simple problem

- Weak form, with homog. boundary conditions

\[ \text{find } u \in H^1_0(\Omega) \text{ such that } \]
\[ a(u, v) = b(v) \quad \forall v \in H^1_0(\Omega) \]

with

\[ a(u, v) = \int_0^L ES \left( \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial x} \right) dx \quad b(v) = \int_0^L f(v) dx \]
Extended Finite Elements

Simple problem

- Discretization: Linear elements, nodal shape functions.

\[ u_h(x) = \sum_i \lambda_i N_i(x) \]
Extended Finite Elements
Simple problem

- By reporting the discrete form of $u$ and $v$ in the weak form, one gets the following linear system:

$$
\begin{bmatrix}
  k_{22} & k_{23} \\
  k_{32} & k_{33}
\end{bmatrix}
\begin{pmatrix}
  \lambda_2 \\
  \lambda_3
\end{pmatrix} = 
\begin{pmatrix}
  f_2 \\
  f_3
\end{pmatrix}
$$

- Here, coefficients $\lambda_1$ and $\lambda_4$ vanish (clamped extremities)
Add two nodes and do the same

This is called « remeshing », it is simple, fast and robust in 1D, less 2D and much less in 3D
Extended Finite Elements
Cut the bar : FEM case

- After discretizing, one gets:

\[
\begin{bmatrix}
  k_{22} & k_{23} & 0 & 0 \\
  k_{32} & k_{33} & 0 & 0 \\
  0 & 0 & k_{44} & k_{45} \\
  0 & 0 & k_{54} & k_{55}
\end{bmatrix}
\begin{bmatrix}
  \lambda_2 \\
  \lambda_3 \\
  \lambda_4 \\
  \lambda_5
\end{bmatrix}
= \begin{bmatrix}
  f_2 \\
  f_3 \\
  f_4 \\
  f_5
\end{bmatrix}
\]

- The two circled parts are independent
- One could solve the linear system separately for each sub-problem
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Cut the bar : FEM case

- The meaning of the DoFs is kept (\( \lambda_i \) means the displacement of node \( i \)).
- There is indeed a discontinuity in the displacement at nodes 3 and 4.
- Nothing changes in the implementation – only the mesh and its topology are modified.
Extended Finite Elements

Cut the bar : X-FEM case

- Now : we don’t change the mesh !
- But one can add/modify shape functions

\[ N_1(x) \quad N_2(x) \quad N_3(x) \quad N_4(x) \]
Case (I) :

\[ N_1(x) \quad N_2^-(x) \quad N_3^+(x) \quad N_4(x) \]
Case (I):

\[ N_1(x) + N_2^-(x) + N_3^+(x) + N_4(x) \]

\[ + N_2^+(x) \]
Case (I):

\[ N_1(x) + N^-_2(x) + N^+_3(x) + N_4(x) \]

\[ N^+_2(x) + N^-_3(x) \]
How to compute the $N_j^+$ from the $N_i$?

- Let’s introduce the Heaviside function:
  
  $$H(s) = \begin{cases} 
  0 & \text{if } s \leq 0 \\
  1 & \text{if } s > 0
  \end{cases}$$

- This is its complement:
  
  $$\bar{H}(s) = \begin{cases} 
  1 & \text{if } s \leq 0 \\
  0 & \text{if } s > 0
  \end{cases}$$

- $s$ is the distance to the cut (here, $s = x - \frac{L}{2}$)
Extended Finite Elements

Cut the bar : X-FEM case (I)

- With these notations, one have:

\[
\begin{align*}
N_i^+(x) &= N_i(x) \cdot H(s) \\
N_i^-(x) &= N_i(x) \cdot \bar{H}(s)
\end{align*}
\]

- One may notice that the partition of unity is preserved
One has to sort the mesh nodes

- Those which have “regular” degrees of freedom go into set $N$
- Those which have modified degrees of freedom go into set $C$

The solution field $u$ is written as:

$$ u(x) = \sum_{i \in N} \lambda_i N_i(x) + \sum_{j \in C} \lambda^+_j N^+_j(x) + \sum_{k \in C} \lambda^-_k N^+_k(x) $$
Extended Finite Elements

Cut the bar : X-FEM case (I)

- **Linear system**
  - We number the DoFs as follows:

\[
\begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
\lambda_1 & \lambda^- & \lambda^- & \lambda^+ & \lambda^+ & \lambda_4
\end{bmatrix}
\]

\[
\begin{bmatrix}
k^-_{22} & k^-_{23} & 0 & 0 \\
k^-_{32} & k^-_{33} & 0 & 0 \\
0 & 0 & k^+_{22} & k^+_{23} \\
0 & 0 & k^+_{32} & k^+_{33}
\end{bmatrix}
\begin{bmatrix}
\lambda^- \\
\lambda^- \\
\lambda^+ \\
\lambda^+
\end{bmatrix}
= \begin{bmatrix}
f^-_2 \\
f^-_3 \\
f^+_2 \\
f^+_3
\end{bmatrix}
\]
Again, we manage to separate the domain in two parts

The signification of the degrees of freedom is partly lost

Some shape functions have to be modified

Two “Heaviside” functions are needed to modify the shape functions
Extended Finite Elements

Cut the bar : X-FEM case (II)

- Without changing the shape functions! (case II)

\[ N_1(x) + N_2(x) + N_3(x) + N_4(x) + N_2^*(x) + N_3^*(x) \]
How to compute the $N_j^*$ from the $N_i$ ?

- Lets introduce the modified Heaviside function:

$$H^*(s) = 2H(s) - 1 = \begin{cases} 
-1 & \text{si } s \leq 0 \\ 
1 & \text{si } s > 0 
\end{cases}$$

- With this notation, one finds that:

$$N_i^*(x) = N_i(x) \cdot H^*(s)$$
Extended Finite Elements

Cut the bar : X-FEM case (II)

- One should again sort the mesh nodes
  - Those which have modified DoFs go into set \( C \)
  - “regular” shape functions are still everywhere (no change with regular FEM in that case)

- The solution field \( u \) is written as:

\[
    u(x) = \sum_{i \in \Omega} \lambda_i N_i(x) + \sum_{j \in C} \lambda_j^* N_j^*(x)
\]
Extended Finite Elements
Cut the bar : X-FEM case (II)

- **Linear system**
  - We number the DoFs as follows:

\[
\begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
\lambda_1 & \lambda_2 & \lambda^*_2 & \lambda_3 & \lambda^*_3 & \lambda_4
\end{bmatrix}
\begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
\lambda_1 & \lambda_2 & \lambda^*_2 & \lambda_3 & \lambda^*_3 & \lambda_4
\end{bmatrix}
\begin{bmatrix}
\begin{bmatrix}
k_{22} & k_{22}^* & k_{23} & k_{23}^* \\
k_{22}^* & k_{23}^* & k_{23} & k_{23}^* \\
k_{32} & k_{32}^* & k_{33} & k_{33}^* \\
k_{32}^* & k_{32}^* & k_{33} & k_{33}^*
\end{bmatrix}
\begin{bmatrix}
\lambda_2 \\
\lambda^*_2 \\
\lambda_3 \\
\lambda^*_3
\end{bmatrix}
= \\
\begin{bmatrix}
\begin{bmatrix}
f_2 \\
f^*_2 \\
f_3 \\
f^*_3
\end{bmatrix}
\end{bmatrix}
\end{bmatrix}
\]
At the matrix level, the two parts are linked
Are there two physically separated parts?
  - Lets assemble the matrix without taking care of
    the boundary conditions, and then determine the
    number of vanishing (singular) eigenvalues of
    this matrix.
    - If there is only one entity, there will be only one
      singular eigenvalue (corresponding to the missing
      Dirichlet BC to get a non singular system)
    - Two singular values → the bar is indeed cut in two,
      and two Dirichlet boundary conditions are needed.
Extended Finite Elements
Cut the bar : X-FEM case (II)

- Case without cut and without BC : typical matrix

\[ K^s = k \cdot \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \]

\[ \det(K^s - \alpha I) = 0 \]

\[ k = \frac{3ES}{L} \]

One eigenvalue vanishes.
Case with a cut and without BC : typical matrix

\[ K^c = k \cdot \begin{bmatrix}
1 & -1 & 1 & 0 & 0 & 0 \\
-1 & 2 & -1 & -1 & 0 & 0 \\
1 & -1 & 2 & 0 & -1 & 0 \\
0 & -1 & 0 & 2 & 1 & -1 \\
0 & 0 & -1 & 1 & 2 & -1 \\
0 & 0 & 0 & -1 & -1 & 1
\end{bmatrix} \]

\[ \det(K^c - \alpha I) = 0 \]

Two eigenvalues vanished : it is OK
The meaning of the degrees of freedom is lost

One keeps classical FE basis functions and add others by enrichment

- A kind of hierachical FE basis is built

Only one enrichment function (simpler !)
Cases (I) and (II) are equivalent (the results are exactly identical)

We indeed have a linear combination between shape functions of (I) and those of (II):

\[ N_2(x) = N_2^+(x) + N_2^-(x) \quad N_3(x) = N_3^+(x) + N_3^-(x) \]

\[ N_2^*(x) = N_2^+(x) - N_2^-(x) \quad N_3^*(x) = N_3^+(x) - N_3^-(x) \]

The case (II) is part of the more theoretical frame – use of a given enrichment function and “constructive” synthesis.
Extended Finite Elements

Definition

- eXtended Finite Element Method
  - It is based on classical FEM basis functions
  - The product between these functions and a given enrichment function $E_k(x)$ is then added
  - These enriched functions are able to represent a specific behavior of the solution field that classical shape functions are unable to represent efficiently. (e.g. a discontinuity)

\[
    u(x) = \sum_{i \in \Omega} \lambda_i N_i(x) + \sum_{k} \sum_{j \in C} \lambda^*_{jk} N_j(x) \cdot E_k(x)
\]
Extended Finite Elements

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Extended Finite Elements
In 2D / 3D

- Case of linear elasticity
- Representation of cracks
- Level-sets
- Crack propagation
A wedge with constrained displacements (linear elast.)

\[
a(\bar{u}, \bar{v}) = \int_{\Omega} \bar{\nabla}^s \bar{u} : \bar{D} : \bar{\nabla}^s \bar{v} \, d\Omega
\]

\[
b(\bar{v}) = \int_{\Omega} \bar{f} \cdot \bar{v} \, d\Omega
\]

find \( \bar{u} \) such that

\[
a(\bar{u}, \bar{v}) = b(\bar{v}) \quad \forall \bar{v}
\]
Extended Finite Elements

2D Example

- Displacements without cut (standard FEM)
  \[ \bar{u}(x) = \sum \lambda_i \cdot \bar{N}_i(x) \]

The \( \bar{N}_i(x) \) are the classical linear shape functions (order 1 Lagrange)
Extended Finite Elements
2D Example

- Lets impose a cut path $\phi$
- Modifications of the function space:

$$
\bar{u}(x) = \sum_{i \in \Omega} \lambda_i \cdot \bar{N}_i(x) \\
+ \sum_{i \in C} \lambda_i^* \cdot \bar{N}_i(x) \cdot H^*(s)
$$

- How to define $H^*(s)$ and the set $C$?
The cutting path may be defined with a "level-set" $\phi$

We have $\phi = \{ x \in \mathbb{R}^3 / lsn(x) = 0 \}$

$lsn(x)$ is the signed distance function (to the interface)

One simply takes:

$s = lsn(x)$

$H^*(s) = H^*(lsn(x))$
Extended Finite Elements

2D Example

- Definition of the enriched degrees of freedom (the set $C$)
  - Those are the nodes of the elements completely cut by $\phi$ (iso-0 of the level-set)
Extended Finite Elements

2D Example

- After assembly and solving the linear system one gets two independent solids (as expected)
- The geometry of $\phi$ may be arbitrary.
- No need of any remeshing
Integration

One need to subdivide elements that are cut by the interface (discontinuous functions to integrate)

On each sub triangle (in red here), a classical Gaussian quadrature is used because the integrand is a polynomial.
Crack modeling

- Historically, this is the first application of the extended finite element method
- The crack propagates, and one does not want to generate a new mesh at each time step
- A crack is in fact an incomplete cutting in the domain
Extended Finite Elements

Cracks

- Geometrical representation of the crack
  - It is not part of the mesh (by definition)
  - Its surface is therefore defined, as before, with a level set $l_{sn}$ that represents the normal distance to the surface.
  - One also needs the location where it stops (on its surface)
    - Crack tip (or front in 3D)
Extended Finite Elements

Cracks

- We make use of another level set $l_{st}(x)$
  - It represents the distance to the crack front (measured tangentially)
  - Both level sets form an orthogonal basis at the crack tip
Extended Finite Elements

Cracks

- The locus of the crack is therefore defined as:
  \[ \phi = \{ x \in \mathbb{R}^3 / \text{lsn}(x) = 0, \text{lst}(x) \leq 0 \} \]

- The enrichment set \( C \) is also modified:

![Diagram of crack and enriched elements]
Extended Finite Elements

Cracks

- The enrichment set $C$ is also modified:
  - Zone of influence of the new shape functions
The enrichment set $C$ is also modified:
- Zone of influence of the new shape functions
The enrichment set $C$ is also modified:

- Zone of influence of the new shape functions
  - Either it cannot cover the crack until its tip or front...
Extended Finite Elements
Cracks

- The enrichment set $C$ is also modified:
  - Zone of influence of the new shape functions
    - Either it cannot cover the crack until its tip or front...
    - or it goes a bit too far
Extended Finite Elements

Cracks

- A special procedure is needed at the crack tip
  - The enrichment function should be discontinuous until the crack tip; continuous beyond.
Extended Finite Elements
Cracks

\[ T(s, t) = \begin{cases} 
0 & \text{if } t \geq 0 \\
H^*(s) & \text{if } t \leq -e \\
\frac{-t H^*(s)}{e} & \text{if } -e < t < 0 
\end{cases} \]

with

\[ \begin{align*}
    s &= lsn(x) \\
    t &= lst(x)
\end{align*} \]

\( e \) could be some elements wide
Extended Finite Elements
Cracks

- Alternate set of enriched elements $C'$
  - It includes every node for which the support is cut (at least partly) by the crack.

\[
\bar{u}(x) = \sum_{i \in \Omega} \lambda_i \cdot \tilde{N}_i(x) + \sum_{i \in C'} \lambda^*_i \cdot \tilde{N}_i(x) \cdot T(t, s)
\]
Extended Finite Elements

Cracks

- Displacements with a crack tip enrichment
Extended Finite Elements

Cracks

- In fact, the form of the exact solution is known at the crack tip
  - Why not use this directly as a crack enrichment function?
  - It is readily available for a crack in an infinite medium → see any fracture mechanics course
A polar basis is defined

\[ r = \sqrt{l_{sn}(x)^2 + l_{st}(x)^2} \]
\[ \theta = \text{arg} \left( \{ l_{st}(x), l_{sn}(x) \} \right) \]
\[ \theta = \arctan \frac{l_{sn}(x)}{l_{st}(x)} \]
Extended Finite Elements

Cracks

- Exact asymptotic fields at the crack tip (crack in an infinite domain)

\[ u_1 = \frac{1}{2\mu} \sqrt{\frac{r}{2\pi}} \left\{ K_1 \cos \frac{\theta}{2} (\kappa - \cos \theta) + K_2 \sin \frac{\theta}{2} (\kappa + 2 + \cos \theta) \right\} \]

\[ u_2 = \frac{1}{2\mu} \sqrt{\frac{r}{2\pi}} \left\{ K_1 \sin \frac{\theta}{2} (\kappa - \sin \theta) + K_2 \cos \frac{\theta}{2} (\kappa - 2 + \cos \theta) \right\} \]

\[ u_3 = \frac{2}{2\mu} \sqrt{\frac{r}{2\pi}} \left\{ K_3 \sin \frac{\theta}{2} \right\} \]

\[ \mu = \frac{E}{2(1+\nu)} \]

\[ \kappa = 3 - 4\nu \]

\( K_1, K_2 \) and \( K_3 \) are constants which depend only on boundary conditions: « stress intensity factors »
Some analytical manipulations lead to:

\[
\begin{align*}
    u_1 &= a_1 \sqrt{r} \sin \frac{\theta}{2} + a_2 \sqrt{r} \cos \frac{\theta}{2} + a_3 \sqrt{r} \sin \frac{\theta}{2} \sin \theta + a_4 \sqrt{r} \cos \frac{\theta}{2} \sin \theta + CL(x) \\
    u_2 &= b_1 \sqrt{r} \sin \frac{\theta}{2} + b_2 \sqrt{r} \cos \frac{\theta}{2} + b_3 \sqrt{r} \sin \frac{\theta}{2} \sin \theta + b_4 \sqrt{r} \cos \frac{\theta}{2} \sin \theta + CL(x) \\
    u_3 &= c_1 \sqrt{r} \sin \frac{\theta}{2} + c_2 \sqrt{r} \cos \frac{\theta}{2} + c_3 \sqrt{r} \sin \frac{\theta}{2} \sin \theta + c_4 \sqrt{r} \cos \frac{\theta}{2} \sin \theta + CL(x)
\end{align*}
\]

One can therefore use only 4 enrichment functions (they span the whole function space):

\[
\begin{align*}
    f_1 &= \sqrt{r} \sin \frac{\theta}{2} \\
    f_3 &= \sqrt{r} \sin \frac{\theta}{2} \sin \theta \\
    f_2 &= \sqrt{r} \cos \frac{\theta}{2} \\
    f_4 &= \sqrt{r} \cos \frac{\theta}{2} \sin \theta
\end{align*}
\]

One may notice that only \( f_1 \) is discontinuous.
Extended Finite Elements

Cracks

- Shape of the enrichment functions in the case of an Irwin crack

\[ f_1 \quad f_2 \]
\[ f_3 \quad f_4 \]
A new function space

\[ \tilde{u}(x) = \sum_{i \in \Omega} \lambda_i \cdot \tilde{N}_i(x) + \sum_{i \in \mathcal{C}} \lambda_i^* \cdot \tilde{N}_i(x) \cdot H^*(s) + \sum_{i \in \mathcal{T}} \sum_{j = 1..4} \lambda_{ij}^j \cdot \tilde{N}_i(x) \cdot f_j(r, \theta) \]

Where to enrich?

- At the crack tip (T), because the rest of the domain is already concerned by the Heaviside enrichment
- The analytical solution used to build the \( f_j(r, \theta) \) is only valid around the crack tip.
Extended Finite Elements

Cracks

- Choice of the nodes to enrich
  - The set $C$ contains nodes for which the support is completely cut by the crack
  - The set $T$ contains the nodes for which the support contains or touches the crack tip
Extended Finite Elements

Cracks

- Displacements with the new crack tip enrichment
If one chooses a good enrichment procedure, one may get a better convergence rate than observed with regular finite elements.
To be able to propagate a crack, it is needend to:

- Perform the assembly of the linear system
- Solve the linear system
- Compute adequate propagation parameters
- Update level-sets $ls_n$ and $ls_t$

Crack propagation obeys to well defined physical laws:

- Fatigue
- Fragile fracture
- etc...
What are the adequate parameters of crack propagation

- Charge coefficients (stress intensity factors) that are linked to the geometry of the problem and the boundary conditions.
  - Intrisic parameters having effects on the material just in front of the crack path.
- Material behaviour with respect to these SIFs: ductile propagation (mild steel) or fragile (glass, cast iron)
- For ductile fracture, one often uses the ratio (number of loading cycle) w.r. to (crack advance)
Extended Finite Elements

Cracks

- Computation of the stress intensity factors
  - Depend only on stress field around the crack
  - J integrals and interaction integrals
  
  \( J = \int_{\Gamma} \left[ \frac{1}{2} \sigma_{ij} \epsilon_{ij} \delta_{1j} - \sigma_{ij} \frac{\partial u_i}{\partial x_1} \right] n_j d\Gamma \)

\[ J^{(1+2)} = \int_{\Gamma} \left[ \frac{1}{2} (\sigma^{(1)}_{ij} + \sigma^{(2)}_{ij})(\epsilon^{(1)}_{ij} + \epsilon^{(2)}_{ij}) \delta_{1j} - (\sigma^{(1)}_{ij} + \sigma^{(2)}_{ij}) \frac{\partial (u_i^{(1)} + u_i^{(2)})}{\partial x_1} \right] n_j d\Gamma = J^{(1)} + J^{(2)} + I^{(1+2)} \]

\[ I^{(1+2)} = \int_{\Gamma} \left[ \sigma^{(1)}_{ij} \epsilon^{(2)}_{ij} \delta_{1j} - \sigma^{(1)}_{ij} \frac{\partial u_i^{(2)}}{\partial x_1} - \sigma^{(2)}_{ij} \frac{\partial u_i^{(1)}}{\partial x_1} \right] n_j d\Gamma \]

\[ I^{(1+2)} = 2 \left( \frac{1 - \nu^2}{E} \right) \left( K^{(1)}_1 K^{(2)}_1 + K^{(1)}_2 K^{(2)}_2 \right) + \frac{1}{\mu} K^{(1)}_3 K^{(2)}_3 \]
Extended Finite Elements
Cracks

- Going from a contour integral to a volume integral (unloaded crack)

\[ I^{(1+2)} = \int_V \frac{\partial q_m}{\partial x_j} \left( \sigma^{(1)}_{kl} \epsilon_{kl} (2) \delta_{mj} - \sigma^{(1)}_{ij} \frac{\partial u_i^{(2)}}{\partial x_m} - \sigma^{(2)}_{ij} \frac{\partial u_i^{(1)}}{\partial x_m} \right) dV \]

One have \( q_m = \alpha \cdot v_m \) and \( \alpha \) is equal to 1 inside the domain and vanishes on the boundary \( \Gamma \). \( v_m \) is the virtual crack propagation speed (norm=1). One interpolates \( \alpha \) on the mesh.
Extended Finite Elements

Cracks

- The interaction integrals allows to compute the stress intensity factors
  - Robust
  - Same good properties as the J- integral
  - See fracture mechanics course(s) for more info.
Extended Finite Elements

Cracks

- **Propagation speed**

Example: Alloys under cyclic loadings

Paris law for the speed of propagation:

\[
\frac{da}{dN} = C \cdot \Delta K^m
\]

<table>
<thead>
<tr>
<th>Alloy</th>
<th>( m )</th>
<th>( C ) (m/cycle)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Steel</td>
<td>3</td>
<td>( 10^{-11} )</td>
</tr>
<tr>
<td>Aluminium</td>
<td>3</td>
<td>( 10^{-12} )</td>
</tr>
<tr>
<td>Nickel</td>
<td>3.3</td>
<td>( 4 \cdot 10^{-12} )</td>
</tr>
<tr>
<td>Titanium</td>
<td>5</td>
<td>( 10^{-11} )</td>
</tr>
</tbody>
</table>
Extended Finite Elements

Cracks

- Direction is along the maximal tangent stess $\sigma_{00}$

$$\begin{align*}
\begin{bmatrix}
\sigma_{00} \\
\sigma_{r0}
\end{bmatrix} &= \frac{K_1}{4\sqrt{2\pi}r} \begin{bmatrix}
3\cos\frac{\theta}{2} + \cos\frac{3\theta}{2} \\
\sin\frac{\theta}{2} + \sin\frac{3\theta}{2}
\end{bmatrix} + \frac{K_2}{4\sqrt{2\pi}r} \begin{bmatrix}
-3\sin\frac{\theta}{2} - 3\sin\frac{3\theta}{2} \\
\cos\frac{\theta}{2} + 3\cos\frac{3\theta}{2}
\end{bmatrix}
\end{align*}$$

$$\frac{\partial \sigma_{00}}{\partial \theta} = 0 \quad \rightarrow \quad \cos\frac{\theta_c}{2} \left[ \frac{1}{2} K_1 \sin \theta_c + \frac{1}{2} K_2 (3\cos \theta_c - 1) \right] = 0$$

$$\theta_c = 2 \arctan \frac{1}{4} \left( \frac{K_1}{K_2} \pm \sqrt{\left( \frac{K_1}{K_2} \right)^2 + 8} \right)$$

- One chooses $\theta_c$ that correspond to a maximal value of $\sigma_{00}$ (in traction)
There exists many algorithms the the essential part is to:

- Conserve the notion of signed distance function at the interface for $\text{lsn}$
- Have an orthonormed frame in the vicinity of the crack tip ($\text{lst,lsn}$)
Extended Finite Elements
Level set update

- Transport of $l_{sn}$ and $l_{st}$
Extended Finite Elements

Level set update

- Rebuilding of $l_{sn}$ and $l_{st}$

\[ dx = l_{st1} - l_{st2} \]
\[ dy = l_{sn1} - l_{sn2} \]
\[ \alpha = \text{atan2}(dy, dx) \]

\[ l_{sn} = -\sin(\alpha) \cdot l_{st2} + \cos(\alpha) \cdot l_{sn2} \]

\[ l_{st} = \cos(\alpha) \cdot l_{st2} + \sin(\alpha) \cdot l_{sn2} \]
Extended Finite Elements
Propagation
Extended Finite Elements
Propagation
Extended Finite Elements
Propagation
Extended Finite Elements
Propagation
Extended Finite Elements
Propagation
Extended Finite Elements
Propagation
Extended Finite Elements
Propagation
Extended Finite Elements
Propagation
Extended Finite Elements

3D Propagation
Extended Finite Elements
3D Propagation
Extended Finite Elements

Tricky points

- Integration
  - One should cut elements along the interface… but one should also change the quadrature or increase the number of quadrature points because the integrand is no more polynomial
Extended Finite Elements

Tricky points

- **Condition number**
  - If the choice of the enriched DoFs is wrongly made, then the condition number will be close to 0 (this yields a singular linear system)
    - Crack goes close to a node → then it goes through it (at least virtually)
  - The enriched shape functions at crack tip may induce a bad condition number (those “look alike”)
    - Use of a specialized preconditionner
Extended Finite Elements

Tricky points

- Condition number
Représentation valable pour les fissure dans les matériaux fragiles
- Fissure mathématiquement représentée par une ligne infiniment fine
- Front de fissure ponctuel, champs infinis
- Lois de propagation basées sur des grandeurs globales (e.g. taux de restitution d'énergie $G$...)

Dans les matériaux ductiles, cela est trop restrictif
Extended Finite Elements
Ductile materials

New properties

- Crack shape absolutely non trivial
- The propagation is made via a damage variable
- The level sets are used to represent at the same time
  - the damage variable $d$
  - the crack front (where $d=1$)
  - the boundary between the damaged zone ($d>0$) and the rest of the domain where the behavior is elastic

- Notion of “Thick” Level Set
Extended Finite Elements

Thick Level Set

Undamaged zone
\[ \varphi \geq 0 \quad d = 0 \]

Damaged zone
\[ \varphi \geq l_c \quad d = 1 \]

 Entirely damaged zone
\[ 0 \leq \varphi \leq l_c \quad 0 \leq d \leq 1 \]

« Crack »
Extended Finite Elements
Thick Level Set

N. Moës, C. Stolz, P.-E. Bernard, and N. Chevaugeon
A level set based model for damage growth: The thick level set
Problems with a jump in the gradient ("dual" variable)
Extended Finite Elements
Bi-material interface

- Thermal transfer model problem
Extended Finite Elements

Bi-material interface

- The interface is represented by the following level-set:
  \[ \phi = \{ x \in \Omega \ / \ ls(x) = 0 \} \]

- This interface can be of complex geometrical shape and/or changing in time
  - Again, no mesh conformity
Extended Finite Elements

Bi-material interface

- Finite element model (again, homogeneous boundary conditions)

Find \( u \in H^1_0(\Omega) \) s.t.

\[
a(u, v) = b(v) \quad \forall \ v \in H^1_0(\Omega)
\]

with

\[
a(u, v) = \int_{\Omega} k(\nabla u \cdot \nabla v) \, d\Omega \quad b(v) = \int_{\Gamma} f(x) \cdot v \, d\Gamma
\]
Extended Finite Elements

Bi-material interface

- We want to be able to represent the right temperature profile along the interface
  - A-A cut: Theoretical temperature profile
Extended Finite Elements

Bi-material interface

- The discontinuity is on the derivative of $T$
- If the interface is exactly on element boundaries, then the discontinuity is naturally belonging to the function space
Extended Finite Elements

Bi-material interface

- The discontinuity is on the derivative of $T$
- If the interface is not exactly on element boundaries, then ...
Extended Finite Elements
Bi-material interface

- This explains the very approximate solution...
Extended Finite Elements

Bi-material interface

- The idea here is to enrich the function space so that the discontinuity (in the gradient) belong to it.

\[ u(x) = \sum_{i \in \Omega} \lambda_i N_i(x) + \sum_{j \in C} \lambda^*_j N_j(x) \cdot F(x) \]

There exists many possibilities. One very simple is using directly the absolute value of the level-set.

\[ F_1(x) = |ls(x)| \]
Extended Finite Elements

Bi-material interface

- Definition of the set $C$ of the enriched nodes
  - This time, the nodes where at least one element of the support are cut must be enriched
  - In particular, if the interface is along edges, there is no enrichment
Extended Finite Elements
Bi-material interface

- Here are some other enrichment functions

\[ F_2(x) = \begin{cases} 
|ls(x)| & \text{in cut elements} \\
1 & \text{else} 
\end{cases} \]

\[ F_3(x) = \sum_i |ls_i| \cdot N_i(x) - \left| \sum_i ls_i \cdot N_i(x) \right| \]

\[ F_1(x) \]

\[ F_2(x) \]

\[ F_3(x) \]
Practically speaking, $F_3(x)$ gives the best results.

On a simple model problem, the functions $F_2(x)$ and $F_3(x)$ are unable to give back the exact solution (which is linear by parts) when the interface does not belong to the mesh, but $F_1(x)$ does.
Extended Finite Elements

Bi-material interface

- Comparison of the solution with the right enrichment function

Solution without enrichment

Solution with enrichment
Extended Finite Elements

Bi-material interface

- Comparison of the solution with the right enrichment function
Extended Finite Elements
Bi-material interface

- Comparison of the gradient

Solution without enrichment

Solution with enrichment
Extended Finite Elements
Bi-material interface

- Comparaison du gradient

Exact solution

Solution with enrichment
Extended Finite Elements

Bi-material interface

- Convergence
Extended Finite Elements

More applications

- Discontinuities in the primal variable
  - Cracks
    - non linearities, plasticity
    - Dynamic effects (fast propagation)
  - Solidification front propagation
    - hydrogels
  - Discontinuities in the derivatives
    - Homogeneization
Extended Finite Elements

More applications

- Applications to other materials
  - Confined plasticity
  - Composites materials
  - Piezoelectric materials
  - Etc...
Extended Finite Elements

More applications

- Direct interfaces with CAD for numerical simulations
  - From an explicit representation to an implicit representation
  - Non conforming boundaries
  - Imposition of boundary conditions
  - Non conforming material interfaces
Extended Finite Elements

More applications

- Applications in explicit dynamics
  - Non conforming geometry → issue with the critical time step
  - Propagation of unstable cracks (change of function space at the crack tip → leads to problems of energy conservation)
Extended Finite Elements

More applications

- Explicit dynamics: case without enrichment
The issue of boundary conditions on implicit volumes
Extended Finite Elements

Goal

- Free the mesh from geometrical constraints
  - Boundaries of the problem
  - And/or interfaces between different materials
Extended Finite Elements
Applications

- Direct use of CAD models for the analysis
  - Mesh generation shall be minimalistic
- Use of “dirty” geometrical data not usually adapted to mesh generation
  - Tomography, biomedical applications
- Mobile interfaces
  - Thermoplastic mold filling
  - Topological shape optimization
- Contact problems in mechanical engineering
Extended Finite Elements

CAD interface

- From a traditional CAD (B-rep) representation ...
Extended Finite Elements

CAD interface

- ... To an implicit representation with level-sets
Extended Finite Elements

Boundary conditions

- How to apply boundary conditions
  - Neumann/natural boundary conditions (e.g. pressure, forces, gradients)
  - Using integration (it is a linear form)
    
    Find \( \bar{u} \) s.t.
    
    \[
    a(\bar{u}, \bar{v}) = b(\bar{v}) \quad \forall \bar{v}
    \]
    
    \[
    a(\bar{u}, \bar{v}) = \int_{\Omega} \bar{N}^s \bar{u} : \bar{D} : \bar{N}^s \bar{v} \, d\Omega
    \]
    
    \[
    b(\bar{v}) = \int_{\Omega} \bar{f} \cdot \bar{v} \, d\Omega + \int_{\Gamma_N} \hat{\bar{f}} \cdot \bar{v} \, d\Gamma_N
    \]
    
  - Beware! The integration is made on a domain \( \Gamma_N \) (or \( \Omega \)) that cut elements in the mesh
How to apply boundary conditions

- Dirichlet/essential boundary conditions (e.g.: displacements, temperature)
  - "standard" FEM elimination of DoFs and adding a contribution in the right hand side
  - Here, the domain $\Gamma_D$ on which to apply this method is non conforming therefore one cannot simply eliminate DoFs - one needs to compute the values to impose at each concerned DoF; so that the "right" Dirichlet BC is obtained on the boundary
Extended Finite Elements

Dirichlet boundary conditions

- Example 1: a simple Laplacian

Find \( u \in V_1 = \{ v \in H^1(\Omega), v|_{\Gamma_D} = u_D \} \) s.t.

\[
\begin{align*}
\forall v \in V_0 &= \{ v \in H^1(\Omega), v|_{\Gamma_D} = 0 \} \\
\int_{\Omega} \nabla u \cdot \nabla v \, d\Omega &= \int_{\Gamma_N} f \cdot v \, d\Gamma \\
\end{align*}
\]

\[
a(u, v) = b(v)
\]
Extended Finite Elements

Dirichlet boundary conditions

- Example 1
  - Dofs which are concerned: those where the support cuts the boundary...
**Extended Finite Elements**

**Dirichlet boundary conditions**

- With only two linear elements?
  - Without Dirichlet B.S. : 4 DoFs, \( u \) has some freedom in the red part
  - If one imposes exactly \( u=0 \) on the boundary ...

\[
\begin{align*}
\frac{u_1}{a_1} &= \frac{u_2}{a_2}; & \frac{u_2}{b_2} &= \frac{u_3}{b_3}; & \frac{u_3}{c_3} &= \frac{u_4}{c_4}
\end{align*}
\]

- How many DoFs left for the red part of the domain?
Concrete example

- Number of available DoFs after imposing *exactly* the Dirichlet B.C.:

  3 !

- The function space is very poor in the elements crossed by the interface, therefore the F.E. solution will be far from accurate.
Extended Finite Elements

Dirichlet boundary conditions

- One cannot impose exactly a Dirichlet B.C. by elimination as long as it is crossing through finite elements!
- For this, an interpolation is preferred and the B.C. must be along element edges.
- This is the reason why Lagrange F.E. are so widely used.
  - (One) solution: the use of lagrange multipliers, see an article of Babuska (1973) - in the bibliography)
Extended Finite Elements
Lagrange multipliers

On wants to minimize \( \pi(u, v) = u^2 + v^2 \)

\[ \delta \pi(u, v) = 2u \delta u + 2v \delta v = 0 \quad \forall \, \delta u, \delta v \]

\[ u = v = 0 \quad \pi(0, 0) = 0 \]

If one sets an additional condition :
\[ g(u, v) = u - v + 2 = 0 \]

Method 1 : elimination of \( v \) :
\[ \pi'(u) = 2u^2 + 4u + 4 \equiv \pi(u, v) \]

\[ \delta \pi'(u) = 4(u + 1) \delta u = 0 \quad \forall \, \delta u \]

\[ u = -1 \quad \pi'(-1) = 2 \rightarrow v = 1 \]

This is the method used just before ...
Method 2: Introduction of an additional variable

\[ \tilde{\pi}(u, v, \lambda) = \pi(u, v) + \lambda g(u, v) = u^2 + v^2 + \lambda (u - v + 2) \]
\[ \delta \tilde{\pi}(u, v, \lambda) = 0 \]
\[ = (2u + \lambda) \delta u + (2v - \lambda) \delta v + (u - v + 2) \delta \lambda \]
\[ \forall \delta u, \delta v, \delta \lambda \]

\[
\begin{cases}
2u + \lambda = 0 \\
2v - \lambda = 0 \\
u - v + 2 = 0
\end{cases} \iff
\begin{cases}
u = -1 \\
v = 1 \\
\lambda = 2
\end{cases}
\]
Extended Finite Elements

Lagrange multipliers

- In finite elements, this gives us:

\[-\Delta u = f \text{ in } \Omega\]
\[u = u_D \text{ on } \Gamma_D\]

of weak form: find \( u \) s.t.

\[\int_{\Omega} \nabla u \cdot \nabla \delta u \, d\Omega = \int_{\Omega} f \, \delta u \, d\Omega \quad \forall \delta u\]

Equivalent to minimize \( F(u) = \frac{1}{2} \int_{\Omega} \nabla u \cdot \nabla u \, d\Omega - \int_{\Omega} f u \, d\Omega \)

if the conditions of Lax-Milgram’s theorem are satisfied.

\[a(u, \delta u) = l(\delta u) \]

\[= \frac{1}{2} a(u, u) - l(u)\]

, for all \( u \) satisfying the B.C. on \( \Gamma_D \). By using Lagrange multipliers for the BC’s, one gets a new functional to minimize:

\[\tilde{F}(u, \lambda) = \frac{1}{2} \int_{\Omega} \nabla u \cdot \nabla u \, d\Omega - \int_{\Gamma_D} \lambda(u - u_D) \, d\Gamma_D - \int_{\Omega} f u \, d\Omega\]
Associated weak form:

\[
\widetilde{F}(u, \lambda) = \frac{1}{2} \int_\Omega \nabla u \cdot \nabla u \, d\Omega - \int_{\Gamma_D} \lambda (u - u_D) \, d\Gamma_D - \int_\Omega fu \, d\Omega
\]

\[
= \frac{1}{2} A(U, U) - L(U)
\]

\[
A(U, U) = (u, \lambda) \cdot \begin{pmatrix} a & b \\ b & 0 \end{pmatrix} \cdot (u, \lambda) = a(u, u) + b(u, \lambda) + b(\lambda, u)
\]

\[
L(U) = l(u) + c(\lambda)
\]

\[
a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, d\Omega
\]

\[
b(u, \lambda) = b(\lambda, u) = -\int_{\Gamma_D} u \cdot \lambda \, d\Gamma_D
\]

\[
l(u) = \int_\Omega fu \, d\Omega
\]

\[
c(\lambda) = -\int_{\Gamma_D} u_D \, d\Gamma_D
\]
Extended Finite Elements

Dirichlet boundary conditions

To simplify notations, let's assign \( v = \delta u, \mu = \delta \lambda \)

Find \( \begin{aligned} u &\in V = \left\{ v \in H^1(\Omega) \right\} \\ \lambda &\in L = \left\{ \mu \in H^{1/2}(\Gamma_D) \right\} \end{aligned} \)

s. t.
\[
\int_{\Omega} \nabla u \cdot \nabla v \, d\Omega - \int_{\Gamma_D} \lambda \cdot v \, d\Gamma = \int_{\Gamma_N} f \cdot v \, d\Gamma \quad \forall v \in V \\
- \int_{\Gamma_D} \mu \cdot u \, d\Gamma = - \int_{\Gamma_D} \mu \cdot u_D \, d\Gamma \quad \forall \mu \in L
\]

The Dirichlet B.C. has been "dualized". This is now a Neumann B.C. on the Lagrange multipliers.
Extended Finite Elements

Dirichlet boundary conditions

- The Lagrange multipliers have a physical meaning
  - In mechanics, it is the force to impose so that the condition on the primal variable is ensured (here, displacements).
  - In our case, it is the gradient of the solution (flux) to impose so that \( u = u_D \) on \( \Gamma_D \).

- We have now a saddle point problem (min-max) – the matrix of the linear system is not definite positive anymore (but still has an inverse and is symmetric)

- Not all solvers are able to handle that – mostly direct solvers and very few iterative solvers.
How to build adequate discrete function spaces

- Find \( u_h \in V_h \subset V = \{ v \in H^1(\Omega) \} \) such that ...
  \( \lambda_h \in L_h \subset L = \{ u \in H^{1/2}(\Gamma_D)' \} \)

- One do not change the primal functional space (for \( u \)). It is the usual finite element space using nodal hat functions

- One need to build a function space for \( \lambda \).
  - Lets try to use an identical function space \( L_h \) for \( \lambda \) (or the restriction to the boundary of such a space... (the trace)
Extended Finite Elements

Dirichlet boundary conditions

- Lets try to use an identical function space $L_h$ for $\lambda$ (or the restriction to the boundary of such a space... (the trace)

- Lets perform a computation. The linear system has the following shape:

\[
\begin{pmatrix}
A_h & B_h^T \\
B_h & 0
\end{pmatrix}
\begin{pmatrix}
u_h \\
\lambda_h
\end{pmatrix}
= \begin{pmatrix}
F_h \\
D_h
\end{pmatrix}
\]

\[
\int_{\Omega} \nabla u \cdot \nabla v \, d\Omega - \int_{\Gamma_D} \lambda \cdot v \, d\Gamma = \int_{\Gamma_N} f \cdot v \, d\Gamma \\
- \int_{\Gamma_D} \mu \cdot u \, d\Gamma = - \int_{\Gamma_D} \mu \cdot u_D \, d\Gamma
\]

\forall v \in V_h

\forall u \in L_h
The we solve it …

- Lagrange multipliers are oscillating.
- The more $h$ (element size) shrinks, the more it oscillates…
Extended Finite Elements
Dirichlet boundary conditions

- What happens?
  - The discrete spaces for $u$ et $\lambda$ are incompatible.
  - Those do not satisfy the Ladyzhenskaya-Babuška-Brezzi (LBB) condition, or inf-sup condition:
    \[
    \inf_{\mu \in L_h} \sup_{u \in V_h} \frac{\int_\Gamma \lambda_h u_h d\Gamma}{\|\lambda\|_{0,\Gamma_D} \|u\|_{1,\Omega}} \geq \alpha > 0
    \]
  - This condition is difficult to check analytically.

Extended Finite Elements

Dirichlet boundary conditions

- Numerical validation of the LBB condition.
  - There exists a “simple” numerical test; see Chapelle, Bathe, 1993 and KJ Bathe 2001 (in the bibliography)
    - One considers a more general problem with an added “stiffness” on the Dirichlet boundary condition (becomes a Robin B.C.) – if \( k \to \infty \), back to a “hard” Dirichlet B.C.

\[
\int_{\Omega} \nabla u \cdot \nabla v \, d\Omega - \int_{\Gamma_D} \lambda \cdot v \, d\Gamma = \int_{\Gamma_N} f \cdot v \, d\Gamma \quad \forall v \in V_h \\
- \int_{\Gamma_D} u \cdot u \, d\Gamma - \int_{\Gamma_D} \frac{1}{k} \lambda \mu \, d\Gamma = -\int_{\Gamma_D} \mu \cdot u \, d\Gamma \quad \forall \mu \in L_h
\]

\[
\begin{pmatrix}
A_h & B_h^T \\
B_h & -\frac{1}{k} M_h
\end{pmatrix}
\begin{pmatrix}
u_h \\
\lambda_h
\end{pmatrix} =
\begin{pmatrix}
F_h \\
D_h
\end{pmatrix}
\]
Extended Finite Elements

Dirichlet boundary conditions

- Chapelle – Bathe numerical test

\[
\begin{pmatrix}
A_h & B_h^T \\
B_h & -\frac{1}{k} M_h
\end{pmatrix}
\begin{pmatrix}
u_h \\
\lambda_h
\end{pmatrix}
=
\begin{pmatrix}
F_h \\
D_h
\end{pmatrix}
\]

- It amounts to check the first non vanishing eigenvalue \((\beta_0)\) of the following eigenproblem:

\[
\frac{1}{h} \left( B_h A_h^{-1} B_h^T \right) W_h = \beta M_h W_h \quad \text{ou} \quad \frac{1}{h} \left( B_h^T M_h^{-1} B_h \right) W_h' = \beta' A_h W_h'
\]

- \(A_h\) must have an inverse

- Does not depend on \(k\)!

- One checks that \(\beta_0\) does not vanish for a sequence of meshes with an increasing density

- Here, \(\alpha = \sqrt{\beta_0}\) (and for \(\alpha\) : see slides before)
Extended Finite Elements

Dirichlet boundary conditions

- Results
  - Two cases:
    - aligned with the mesh
    - non conforming
  - The second case does not work at all.

![Graph showing results of infsup (boundary fitted) and infsup (naive algo) with 1/h on the x-axis and different scales on the y-axis ranging from 1e-04 to 1 on various scales.](image)
What we have are incompatibles functional spaces...

- The space for the Lagrange multipliers is way too “rich” with respect to the one for the primal variable.
- It amounts to impose exactly the Dirichlet B.C., which has been already shown to be a bad idea.

→ We have to “decimate” $L_h$
Extended Finite Elements

Dirichlet boundary conditions

- From the mesh of the interface, take each node and put it in a set \( N \).
- If a node of \( N \) is also part of the mesh, mark it as Vital (set \( V \)), and delete it from \( N \).
- Take each edge incident to \( N \) and count each intersecting edge going from end nodes with the interface.
- Sort \( N \). The sorting key is the number defined above (smallest first).

Loop over the sorted set \( N \), take \( n_i \):
  - Take the end nodes of \( n_i \), and from those, the connected nodes in \( N \) (may be many).
  - If \( n_i \) is not yet \( NV \) (non vital), mark it as Vital (V) and all the other connected nodes as (NV).
- EndLoop.
What remains,

- An approximately uniform distribution of nodes

- The density is same as the initial mesh (2D here, 3D in general)

- Works in 3D!
Extended Finite Elements

Dirichlet boundary conditions

Result of the decimation

Projection of 3D nodes
Extended Finite Elements

Dirichlet boundary conditions

Result of the decimation

Projection of 3D nodes
Extended Finite Elements

Dirichlet boundary conditions

- How to build shape functions from this?
  - Directly on the interface?

- Works...

  ... only in 2D !!!
Extended Finite Elements

Dirichlet boundary conditions

- In 3D: one would have to build a triangulation of the set of nodes $V$

What about:

- Curvy interfaces
- Discrepancy (non-conformity) btw. triangulations
  - Integration problems
- So we must find a better way in 3D...
Extended Finite Elements

Dirichlet boundary conditions

- Another solution

  - Lets take the trace of volume shape functions – but there are too many!

    - One will combine SFs. (linear combinations) for each V-node

- At some places, a volume SF may be linked to more than one V-node.

- There is room for freedom: 100% with the green, or 100% with the red or whatever combination such that the sum is 100% (to keep “partition of unity”)
Extended Finite Elements

Dirichlet boundary conditions

- Advantages of using trace shape function for Lagrange multipliers
  - Easy integration
  - Compact shape functions
  - Partition of unity on the interface
  - Same algorithm in 3D and 2D
  - Good numerical results? See what’s follow!
Extended Finite Elements

Dirichlet boundary conditions

2D

3D
Extended Finite Elements

Dirichlet boundary conditions
Extended Finite Elements

Dirichlet boundary conditions
Extended Finite Elements

Dirichlet boundary conditions

- LM error (boundary fitted)
- LM error (new algo PU)
- LM error (naive algo) \( \times 10^{-4} \)
- LM error (old algo)
Extended Finite Elements

Dirichlet boundary conditions
Extended Finite Elements
Dirichlet boundary conditions

Composites: perfect glueing

Imperfect glueing
Extended Finite Elements

Dirichlet boundary conditions
Extended Finite Elements

Dirichlet boundary conditions

Diagram showing energy error for different algorithms as a function of $1/h$. The graph compares:
- Energy error (old algo)
- Energy error (new algo PU)
- Energy error (naive algo)
Extended Finite Elements

Dirichlet boundary conditions

- LM error (old algo)
- LM error (new algo PU)
- LM error (naive algo) ($\times 10^{-4}$)

1/h vs. $10^{-k}$
Extended Finite Elements
Cad Interface

- From a traditional CAD (B-rep) representation ...
Extended Finite Elements
CAD interface

- To an implicit representation and F.E. computation (here, no mesh generation steps, only mesh cutting ... )
Extended Finite Elements

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Nota :

IJNME = International journal for numerical methods in engineering (Wiley)
CMAME = Computer methods in applied mechanics and engineering (Elsevier)
FEAD = Finite element in analysis and design (Elsevier)
JCP = Journal of computational physics (Elsevier)